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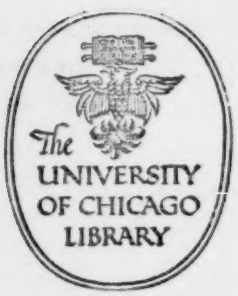

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CONTACT TRANSFORMATIONS LINEAR IN x, y, z; APPLICATIONS TO EQUILONG TRANSFORMATIONS.

By B. H. Brown.

We consider transformations of the form

$$
\begin{aligned}
X &= a_{11}x + a_{12}y + a_{13}z + \lambda, \\
Y &= a_{21}x + a_{22}y + a_{23}z + \mu, \\
Z &= a_{31}x + a_{32}y + a_{33}z + \nu, \\
P &= -\frac{pA_{11} + qA_{12} - A_{13}}{pA_{31} + qA_{32} - A_{33}}, \\
Q &= -\frac{pA_{21} + qA_{22} - A_{23}}{pA_{31} + qA_{32} - A_{33}};
\end{aligned} \tag{1}
$$

where:

(a) a_{ij}, λ, μ, ν are functions of p and q;

(b) A_{ij} represents the cofactor of a_{ij};

(c) $|a_{ij}| \not\equiv 0$, hence $J \not\equiv 0$;

(d) no one of the three expressions $pA_{13} + qA_{23} - A_{33}$ vanishes.

Every such transformation carries any aggregate of ∞^2 planar elements with the same plane into a similar aggregate; but this is not necessarily a plane transformation. A transformation which carries planes into planes carries the point of intersection of three infinitely near planes into the point of intersection of the transformed planes; that is, a plane transformation is necessarily a contact transformation. Not all plane transformations are of type (1), but all contact transformations of type (1) are plane transformations.

For a contact transformation it is necessary and sufficient that

$$
dZ - PdX - QdY \equiv \varrho(dz - p\,dx - q\,dy); \tag{2}
$$

where $\varrho(x, y, z, p, q) \not\equiv 0$. Expanding (2) we have

$$
\begin{aligned}
Z - PX_x - QY_x &\equiv -p\varrho, \\
Z - PX_y - QY_y &\equiv -q\varrho, \\
Z - PX_z - QY_z &\equiv \varrho, \\
Z - PX_p - QY_p &\equiv 0, \\
Z - PX_q - QY_q &\equiv 0.
\end{aligned} \tag{3}
$$

The first three equations of (3) serve merely to calculate the P and Q, given in (1). It will be observed, that P, Q, and -1 are proportional to the two-rowed minors of the matrix

$$\left\| \begin{matrix} pa_{13}+a_{11} & qa_{13}+a_{12} \\ pa_{23}+a_{21} & qa_{23}+a_{22} \\ pa_{33}+a_{31} & qa_{33}+a_{32} \end{matrix} \right\|. \tag{4}$$

The last two equations of (3) are then necessary and sufficient. The expansion of the fourth equation of (3) gives

$$\begin{aligned} &P(a_{11_p}x+a_{12_p}y+a_{13_p}z+\lambda_p) \\ &+Q(a_{21_p}x+a_{22_p}y+a_{23_p}z+\mu_p) \\ &-(a_{31_p}x+a_{32_p}y+a_{33_p}z+\nu_p)\equiv 0. \end{aligned} \tag{5}$$

Equation (5) is explicit in x, y, and z and yields the four equations

$$\begin{aligned} Pa_{11_p}+Qa_{21_p}-a_{31_p} &= 0, \\ Pa_{12_p}+Qa_{22_p}-a_{32_p} &= 0, \\ Pa_{13_p}+Qa_{23_p}-a_{33_p} &= 0, \\ P\lambda_p \ +Q\mu_p \ -\nu_p \ &= 0. \end{aligned} \tag{6}$$

From equations (6) and the similar set involving partial derivatives with regard to q, and from the fact that none of the two-rowed minors of the matrix (4) vanish, it follows that:

THEOREM. *A necessary and sufficient condition that a transformation of type* (1) *be a contact transformation is the vanishing of all the three-rowed minors of the ten by three matrix*

$$\left\| \begin{matrix} pa_{13}+a_{11} & qa_{13}+a_{12} & a_{11_p} & a_{12_p} & a_{13_p} & \lambda_p & a_{11_q} & a_{12_q} & a_{13_q} & \lambda_q \\ pa_{23}+a_{21} & qa_{23}+a_{22} & a_{21_p} & a_{22_p} & a_{23_p} & \mu_p & a_{21_q} & a_{22_q} & a_{23_q} & \mu_q \\ pa_{33}+a_{31} & qa_{33}+a_{32} & a_{31_p} & a_{32_p} & a_{33_p} & \nu_p & a_{31_q} & a_{32_q} & a_{33_q} & \nu_q \end{matrix} \right\|. \tag{7}$$

Equilong transformations are defined as analytic transformations of oriented planes which carry a plane into a plane and preserve invariant the distance between the points of contact of a plane with any two envelopes which it touches.

Equilong transformations form a subgroup of the contact transformations (1). To show this, it is merely necessary to prove that they are linear in x, y, and z when expressed in terms of Lie planar elements. The equations* of (direct) equilong transformations, once-extended, are

$$U = U(u), \qquad V = V(v), \qquad W = \sqrt{U'V'}\,w + F(U, V),$$

$$\frac{\partial W}{\partial U} = \frac{V'^{\frac{1}{2}}U''}{2U'^{\frac{3}{2}}}w + \frac{\partial w}{\partial u}\sqrt{\frac{V'}{U'}} + F_U, \tag{8}$$

$$\frac{\partial W}{\partial V} = \frac{U'^{\frac{1}{2}}V''}{2V'^{\frac{3}{2}}}w + \frac{\partial w}{\partial v}\sqrt{\frac{U'}{V'}} + F_V.$$

The coördinates here used are Bonnet† oriented tangential coördinates defined by the equation

$$(u+v)x - i(u-v)y + (1-uv)z = w. \tag{9}$$

This equation together with

$$x - iy - vz = \frac{\partial w}{\partial u}, \tag{9}$$

$$x + iy - uz = \frac{\partial w}{\partial v}$$

gives on solution the coördinates (x, y, z) of the point of contact of a plane and any envelope.

The coördinates (X, Y, Z) of the point of contact of the transformed plane with the transformed envelope are linear in W, $\partial W/\partial U$, $\partial W/\partial V$, which are themselves linear in w, $\partial w/\partial u$, $\partial w/\partial v$ by (8), and therefore in x, y, and z since u and v, and therefore U and V, are functions of p and q only. Hence equilong transformations are of type (1) and we could actually obtain the Lie form of all such transformations from (8) by differentiation and substitution. The inverse process is extremely difficult. We proceed to find necessary and sufficient conditions on the functions a_{ij} of (1) in order that these transformations be equilong.

An equilong transformation will carry two planar elements with coincident planes into two planar elements with coincident planes. For the points of

* Blaschke, Archiv der Math. u. Phys., Bd. 16, 1910. The lower sign in the equation for W is incorrect, and should be omitted.

† Liouville, 2^{e} série, t. 5.

such planar elements, and not in general for others, square of distance is invariant. We must impose the condition

$$\sum (X_1 - X_2)^2 \equiv \sum (x_1 - x_2)^2 \tag{10}$$

by virtue of

$$(x_1 - x_2)\,p + (y_1 - y_2)\,q - (z_1 - z_2) = 0. \tag{11}$$

Expanding equation (10) we have

$$\begin{aligned} &A(x_1-x_2)^2 + B(y_1-y_2)^2 + C(z_1-z_2)^2 \\ &+2D(x_1-x_2)(y_1-y_2) + 2E(y_1-y_2)(z_1-z_2) + 2F(z_1-z_2)(x_1-x_2) \equiv 0 \end{aligned} \tag{12}$$

by virtue of (11), where the terms $A, \cdots, F$ are

$$A = a_{11}^2 + a_{21}^2 + a_{31}^2 - 1, \qquad D = a_{11}\,a_{12} + a_{21}\,a_{22} + a_{31}\,a_{32}, \text{ etc.}$$

We may eliminate either $(x_1 - x_2)$, $(y_1 - y_2)$ or $(z_1 - z_2)$ between (11) and (12). In any case we shall have three coefficients to equate to zero. These three eliminations give the equations

$$\begin{cases} Cp^2 + 2Fp + A = 0, \\ Cq^2 + 2Eq + B = 0, \\ Cpq + Ep + Fq + D = 0, \end{cases}$$

$$\begin{cases} Aq^2 - 2Dpq + Bp^2 = 0, \\ Cp^2 + 2Fp + A = 0, \\ Ep^2 - Fpq + Dp - Aq = 0, \end{cases} \tag{13}$$

$$\begin{cases} Aq^2 - 2Dpq + Bp^2 = 0, \\ Cq^2 + 2Eq + B = 0, \\ Fq^2 - Epq - Bp + Dq = 0. \end{cases}$$

From (13), by simple combinations, we derive

$$\begin{aligned} p(E^2 - BC) + q(DC - FE) - (FB - DE) &= 0, \\ p(DC - FE) + q(F^2 - AC) - (AE - FD) &= 0, \\ p(FB - DE) + q(AE - FD) - (D^2 - AB) &= 0. \end{aligned} \tag{14}$$

The determinant of (14) vanishes, hence

$$(15) \qquad \begin{vmatrix} A & D & F \\ D & B & E \\ F & E & C \end{vmatrix} = 0;$$

for the coefficients in (14) are cofactors of the elements of (15), which is therefore a necessary condition. In general any set of (13) is sufficient; or (15), one of (14), and one of (13).

It is interesting to note that if $|a_{ij}|$ is symmetric,

$$\begin{vmatrix} a_{11}+1 & a_{12} & a_{13} \\ a_{21} & a_{22}+1 & a_{23} \\ a_{31} & a_{32} & a_{33}+1 \end{vmatrix} \cdot \begin{vmatrix} a_{11}-1 & a_{12} & a_{13} \\ a_{21} & a_{22}-1 & a_{23} \\ a_{31} & a_{32} & a_{33}-1 \end{vmatrix} = \begin{vmatrix} A & D & F \\ D & B & E \\ F & E & C \end{vmatrix} = 0;$$

hence one of the simpler determinants is zero.

As an illustration of the general theory let us consider the transformation of Bonnet*

$$X = x+pz, \qquad Y = y+qz, \qquad Z = izR,$$

where $R = \sqrt{1+p^2+q^2}$, which carries minimal surfaces into Bonnet surfaces. In this case the ten by three matrix is

$$\begin{Vmatrix} p^2+1 & pq & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ pq & q^2+1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ ipR & iqR & 0 & 0 & \frac{ip}{R} & 0 & 0 & 0 & \frac{iq}{R} & 0 \end{Vmatrix};$$

and we have

$$A = 0, \qquad B = 0, \qquad C = -2,$$
$$D = 0, \qquad E = q, \qquad F = p.$$

It will be seen that equations (13), (14), (15) are satisfied, hence Bonnet's transformation is equilong.

It will be noted that for Bonnet's transformation the rank of (15) is two. This is the general case. We have found no geometric significance attached

* Comptes Rendus, 1856, t. 42, p. 485.

to the rank one. If (15) is of rank zero, that is, if (10) holds independently of (11) the most general transformation is of the form

$$X = x + \lambda(p, q), \quad Y = y + \mu(p, q), \quad Z = z + \nu(p, q), \quad P = p, \quad Q = q, \tag{16}$$

plus rotations and a reflection in the origin; where λ, μ, ν satisfy

$$\begin{aligned} p\lambda_p + q\mu_p - \nu_p &= 0, \\ p\lambda_q + q\mu_q - \nu_q &= 0. \end{aligned} \tag{17}$$

This is essentially proved by Blaschke* by a consideration of the geometrical properties of the equilong transformations which keep invariant the distance between parallel planes. Such transformations transform rigidly every aggregate of ∞^3 planar elements with parallel planes. This last property characterizes the transformations (1) for which (15) has the rank zero. This may also be shown by a direct analytic method, the functions a_{ij} reducing to constants with orthogonal determinant and equations (17) coming directly from (7).

It is easy to verify that a necessary and sufficient condition that the parameters u and v of the surface

$$x = \lambda(u, v), \quad y = \mu(u, v), \quad z = \nu(u, v), \tag{18}$$

have the geometric significance of p and q is that equations (17) hold. This is to be expected, for by a theorem due to Study†, the transformations (17) are sufficient to transform a point (e. g. the origin) into any surface (e. g. that indicated by (18)).

The orientation of a normal to a surface at a point is wholly undetermined by the parameters p and q. When we change from the equation of an oriented surface in oriented tangential coördinates to a form (18) with p and q as parameters, the orientation is lost, but generally the expression $\sqrt{1+p^2+q^2}$ occurs frequently and the selection of one of the two functions indicated by this radical determines the orientation. It may happen that this radical does not occur, and we should expect this to be the case when

* Loc. cit., p. 188.

† Sitzungsberichte der Niederrheinischen Gesellschaft für Natur- und Heilkunde, Dez. 5 1904. Study's theorem is incorrect in that he states there is only one such transformation there are always two distinct transformations.

the oriented surface is one-sided.* We may therefore expect to find among the rational solutions of (17), functions λ, μ, ν which yield a parametric representation of well-known one-sided surfaces, although not necessarily one-sided in the real domain. This is actually the case. The solution

$$\lambda = q, \qquad \mu = p, \qquad \nu = pq$$

gives the hyperbolic paraboloid $z = xy$, one-sided in the complex domain; while

$$\lambda = \frac{2pq^2}{(p^2+q^2)^2}, \qquad \mu = -\frac{2p^2q}{(p^2+q^2)^2}, \qquad \nu = \frac{q^2}{p^2+q^2}$$

gives the ruled cubic $z = \frac{x^2}{x^2+y^2}$, due to H. S. Smith, which is one-sided in the real domain.

* For a full bibliography on the subject of one-sided surfaces, see Brocard, L'Intermédiaire des Math., vol. 8, 1901, p. 294.

HARVARD UNIVERSITY.

ON THE TRIGONOMETRIC REPRESENTATION OF AN ILL-DEFINED FUNCTION.*

By Dunham Jackson.

1. **Introduction.** In the theory of functions of a real variable x, it is commonly supposed that there is a single definite value of the function for each value of x; or, if the function is multiple-valued, each determination is considered by itself, as belonging to a particular branch. In the case of a function known only from physical observation, however, it is likely to happen that a number of more or less discordant determinations of the dependent variable y are found corresponding to the same value, or to nearly equal values, of the independent variable, and these are not determinations of distinct branches of the function, but are approximations to a single quantity, or representative values of a quantity which is not absolutely determinate in an ideal sense at all. If the observations are confined to a small number of different values of the independent variable, it is natural to adopt some sort of average as the best attainable determination of the function for each of the particular values of x in question, and to fill in the intervals between by some process of interpolation. When a large number of values of x are concerned, however, and these are themselves distributed more or less at random, it is possible to idealize the problem in a different way. It may be supposed namely that, for an arbitrarily given x, *any* value of y, or any value within bounds, is a possible determination of the function in question, but that some values of y are more likely to be found than others, that is, are asymptotically of more frequent occurrence in a long series of observations, to an extent indicated by a *weight* which is a function of x and y, essentially positive or zero. That is, a function of two variables, $w(x, y)$, having the last-named property, can be regarded as constituting a sort of function of the single variable x, which is blurred or out of focus, and can be made distinct only by a more or less arbitrary averaging process.

The following discussion is a development of the idea thus vaguely expressed,† in the form of a study of certain properties of non-negative

* Presented to the American Mathematical Society, December 30, 1920.

† The function $w(x, y)$ is of course merely a frequency-function, in the sense of the theory of statistics; but insistence on the technical term in the present paper might seem to imply a closer connection with the ordinary content of that theory than actually exists. No special knowledge of statistical theory is presupposed on the part of the reader, and no claims are advanced as to the immediate practical availability of the results obtained. Some of the facts presented in the early pages are undoubtedly well known, at least in substance, but the writer does not recall any treatment of the subject from the point of view taken here.

functions $w(x, y)$. It will be convenient to impose certain requirements of continuity, by which the discussion is perhaps removed from the realm of immediate practical applicability; but even so, the results obtained may have some qualitative significance.

2. **The arithmetical mean.** In all that follows, it will be assumed that $w(x, y)$ is a continuous function of both its arguments together, for all values of x in an interval $a \leqq x \leqq b$ (which may be infinite in extent), and for all real values of y. It will be assumed that $w(x, y) \geqq 0$ throughout the region considered, and that for each value of x there are values of y for which $w > 0$. Finally, it will be supposed for simplicity that $w = 0$ identically for all values of y outside a finite interval $c < y < d$, in other words, that no values of y outside this interval come into consideration as possible determinations of the observed quantity.

One way, perhaps the most natural way, of obtaining from $w(x, y)$ a single definite value of y to be associated with each value of x, is to take the arithmetical mean of a set of values distributed according to the weights w, that is, to define

$$\bar{y} = F(x) = I_1(x)/I_2(x), \tag{1}$$

where

$$I_1(x) = \int_c^d y\, w(x, y)\, dy, \qquad I_2(x) = \int_c^d w(x, y)\, dy.$$

Under the hypotheses stated, both integrals are continuous functions of x, and $I_2(x)$ is always different from zero, so that $F(x)$ is continuous.

The same values of $\bar{y}$ can be defined, according to a familiar process, as the solution of a problem in minima. Let

$$H(z, x) = \int_c^d (z - y)^2\, w(x, y)\, dy.$$

For a fixed value of x, this quantity increases without limit as z becomes positively or negatively infinite, and as its dependence on z is continuous, it must have a minimum. Moreover, the derivative of the integrand with regard to z is a continuous function of all its arguments, and the minimum can therefore be found by direct differentiation under the integral sign. The condition determining z is that

$$2\int_c^d (z - y)\, w(x, y)\, dy = 0, \qquad z = I_1/I_2, \tag{2}$$

where I_1 and I_2 have the meanings already indicated.

3. Trigonometric representation by the method of least squares. Let it be supposed now that $w(x, y)$ satisfies the hypotheses for all real values of x, and has the period 2π with regard to this variable. The problem may be proposed of determining, among all trigonometric sums $T_n(x)$ of the nth order,* the one for which the value of the iterated integral

$$\int_0^{2\pi} dx \int_c^d w(x, y)\,[y - T_n(x)]^2\,dy \tag{3}$$

is a minimum. This formulation does not in any way presuppose the definition of an average value of y, as a single-valued function of x, whether by the method of the preceding section or any other; the problem is expressed in terms of the given function $w(x, y)$ itself.

It has first to be recognized that the problem of minimizing the integral has a unique solution. Most of the details that go to make up the proof have been set forth so many times in other connections that it would perhaps be unnecessarily tedious to give them at length here. In outline, the argument is as follows. If any coefficient of $T_n(x)$ were large, the value of $T_n(x)$ itself would be large† for some value of x. If $T_n(x)$ were large for any value of x, it would remain large throughout an interval of appreciable extent,‡ and the value of the integral would be large.§ So the coefficients in all sums $T_n(x)$ which yield small values of the integral must be bounded, and there must be at least one determination of them which makes the integral a minimum. If two different sums $T_n(x)$ give the same value to the integral, their average gives it a smaller value, and so the determination of the sum giving the minimum must be unique.‖

While the minimum problem now under discussion is at the outset proposed independently of the one in the preceding section, the two problems are as a matter of fact intimately connected. Let $z = \overline{y} = F(x)$ be the average value of y corresponding to an arbitrary value of x, as defined by (1) and (2). If $y - T_n(x)$ is broken up into the components

$$y - T_n(x) = (y - z) + [z - T_n(x)],$$

* The words "of the nth order" will be understood to mean "of the nth order at most".

† Cf., e. g., D. Jackson, On functions of closest approximation, Transactions of the Amer. Math. Soc., vol. 22 (1921), pp. 117–128; § 2.

‡ Cf., e. g., D. Jackson, Note on an ambiguous case of approximation, Trans. Amer. Math. Soc., vol. 25 (1923), pp. 333–337; first part of § 2.

§ The truth of this clause is most conveniently established by a method somewhat different from that of the passage just cited, but is fairly obvious when the preceding statements are granted.

‖ Cf., e. g., D. Jackson, first paper cited above, § 6.

ist is seen that the integral (3) can be written in the form

$$\int_0^{2\pi} dx \int_c^d w(x, y)\,(y-z)^2\,dy + 2\int_0^{2\pi} dx \int_c^d w(x, y)\,(y-z)\,[z - T_n(x)]\,dy$$
$$+ \int_0^{2\pi} dx \int_c^d w(x, y)\,[z - T_n(x)]^2\,dy = J_1 + J_2 + J_3.$$

Of these three integrals, the first, J_1, is independent of $T_n(x)$. In the second, the factor $z - T_n(x)$ is independent of y, so that it can be taken outside the y-integral and reserved for the integration with regard to x, and the integral actually to be evaluated with y as variable of integration is

$$\int_c^d w(x, y)\,(y-z)\,dy,$$

which is zero, by the relation (2). That is,

$$J_2 = 0.$$

In the third integral, let

$$\varrho(x) = \int_c^d w(x, y)\,dy;$$

then the whole integral J_3 reduces to

$$J_3 = \int_0^{2\pi} \varrho(x)\,[F(x) - T_n(x)]^2\,dx.$$

As far as the determination of $T_n(x)$ is concerned, then, the problem of minimizing the integral (3) is the same* as the problem of minimizing the integral J_3, in the form last indicated.

If it happens that $\varrho(x)$ is a constant, in other words, that the observations corresponding to the weight-function $w(x, y)$ are evenly distributed with regard to x, then $T_n(x)$ will be simply the partial sum of the Fourier series for $F(x)$. Otherwise, the sum $T_n(x)$ comes within the scope of a recent paper of the writer† on "weighted trigonometric series." In either case,

* It will be recognized that the analysis of this paragraph is suggested by a well-known theorem concerning the moment of inertia of a system about its center of gravity.

† D. Jackson, Note on the convergence of weighted trigonometric series, Bull. Amer. Math. Soc., vol. 29 (1923), pp. 259–263.

if n is allowed to become infinite, $T_n(x)$ will converge uniformly to the value $F(x)$, under favorable circumstances, and it is easy to state sufficient conditions for convergence, at least in terms of properties of the average-function $F(x)$. The question of the hypotheses to be imposed on $w(x, y)$ will not be taken up in detail here, beyond the remark that if w has a first derivative with regard to x which is continuous in both variables, then $F(x)$, as represented by (1), will have a continuous derivative, and the convergence will be assured, with something to spare.

4. **The mean as defined by the method of least mth powers.** The definition adopted in § 2 for the average value of y corresponding to an arbitrary value of x is not the only conceivable one. It would be possible, for example, to define $z = F_m(x)$ as the value of z which minimizes the integral

$$H_m(z, x) = \int_c^d |z - y|^m w(x, y)\, dy,$$

where the constant exponent m will for the sake of simplicity be taken as greater than 1, but is otherwise arbitrary. In this notation, the corresponding integral of § 2 would be denoted by $H_2(z, x)$.

The integral H_m is a continuous function of its arguments, is always positive, and increases without limit as z goes to infinity in either direction, for any fixed value of x (in fact uniformly for all values of x, but that is immaterial). So there is for each x at least one z which makes it a minimum. The uniqueness of the determination of z can be proved by the usual device of substituting the average of any two values of z supposed to give the same minimum, the result being a smaller value of H_m, contrary to the hypothesis.

The quantity $|z - y|^m$ has a derivative with regard to z, $\pm m|z - y|^{m-1}$, the sign being $+$ for $z \geqq y$ and $-$ for $z \leqq y$. This derivative is continuous for all values of y and z, since $m > 1$. Hence the expression for H_m can be differentiated under the sign of integration:

$$(4) \qquad \frac{\partial}{\partial z} H_m(z, x) = H_{mz}(z, x) = m \int_c^d \pm |z - y|^{m-1} w(x, y)\, dy,$$

the ambiguous sign being taken as just indicated, and the new integral is again a continuous function of z and x. If $z_1 > z_2$,

$$|z_1 - y|^{m-1} \operatorname{sgn}(z_1 - y) > |z_2 - y|^{m-1} \operatorname{sgn}(z_2 - y)$$

for all values of y, and hence

$$H_{mz}(z_1, x) > H_{mz}(z_2, x)$$

for all values of x. Thus it is recognized once more that the value of $z = F_m(x)$ corresponding to the minimum of H_m is unique, since the vanishing of H_{mz} is a necessary condition for a minimum. Further than this, however, it can be seen now that $F_m(x)$ is continuous. For if

$$H_{mz}(z_0, x_0) = 0,$$

and if ϵ is an arbitrarily small positive quantity, then

$$H_{mz}(z_0 + \epsilon, x_0) > 0, \qquad H_{mz}(z_0 - \epsilon, x_0) < 0,$$

and, by continuity,

$$H_{mz}(z_0 + \epsilon, x) > 0, \qquad H_{mz}(z_0 - \epsilon, x) < 0,$$

for all values of x in the neighborhood of x_0, so that H_{mz} must vanish between $z_0 - \epsilon$ and $z_0 + \epsilon$ for all such values of x. But the value of z for which H_{mz} vanishes is $F_m(x)$, and so

$$F_m(x_0) - \epsilon < F_m(x) < F_m(x_0) + \epsilon$$

for x sufficiently near to x_0.

In preparation for the reasoning of the following section, it should be pointed out explicitly that

$$c < F_m(x) < d$$

for all values of x. This is fairly obvious from the definition of $F_m(x)$, and is further proved by the fact that that H_{mz}, as represented by (4), can not vanish unless the value of z is such that $z - y$ changes sign as y passes over the interval of integration.

5. **Trigonometric representation by the method of least mth powers.** The problem of trigonometric representation discussed in § 3 can be generalized by inquiring after the trigonometric sum $T_n(x)$ of given order n which minimizes the integral

$$\int_0^{2\pi} dx \int_c^d w(x, y) \, |y - T_n(x)|^m \, dy,$$

the exponent m being once more a given constant greater than 1. By reasoning analogous to that outlined at the corresponding stage in § 3 it is seen that this requirement defines a single sum $T_{mn}(x)$ for each value of m.

On the other hand, the analysis by which it was shown that $T_{2n}(x)$ is at the same time the sum which gives the best approximation to the single-valued function $F_2(x)$, in the sense of the method of least squares, breaks down if the attempt is made to extend it to a general exponent m, and no immediate relation between $T_{mn}(x)$ and $F_m(x)$ is apparent in the general case. *It is all the more noteworthy, then, that $T_{mn}(x)$ still converges uniformly to the value $F_m(x)$ when n becomes infinite,* under fairly general hypotheses.

For the sake of a material simplification in the work, it will be assumed from now on that $m > 2$. The case $m = 2$ has been treated in an earlier section; values of m between 1 and 2 are left in abeyance.

An immediate consequence of the new hypothesis is that the function H_m of § 4 has a continuous second derivative with regard to z,

$$\frac{\partial^2}{\partial z^2} H_m(z, x) = m(m-1) \int_c^d |z-y|^{m-2} w(x, y)\, dy,$$

which is always positive. If $w(x, y)$ has a derivative with regard to x which is a continuous function of x and y, it will follow from well-known theorems on implicit functions that the equation $H_{mz}(z, x) = 0$ defines $z = F_m(x)$ as a function possessing a continuous first derivative with regard to x, because H_{mz} itself will have continuous first derivatives with regard to x and with regard to z, the latter being always different from zero. The condition presently to be imposed on $F_m(x)$, as a basis for the convergence proof, will then be more than satisfied.

Throughout the rest of the section, m will be held fast, and will be omitted as a subscript, for the sake of simplicity, $T_{mn}(x)$ and $F_m(x)$ being denoted merely by $T_n(x)$ and $F(x)$.

Let $\omega(\delta)$ be the modulus of continuity* of $F(x)$, that is, the maximum of $|F(x') - F(x'')|$ for $|x' - x''| \leqq \delta$. To restrict attention from the start to the particular hypothesis which will be found sufficient for convergence, let it be supposed that

$$\lim_{\delta = 0} \omega(\delta)/\sqrt{\delta} = 0.$$

This relation is surely satisfied, for example, if $F(x)$ has a continuous derivative, as in the second paragraph preceding, because then $\omega(\delta) \leqq M\delta$, if M is the maximum of $|F'(x)|$.

Under the condition just stated, there will exist,† for each positive integral

* For the name, cf. de la Vallée Poussin, Leçons sur l'approximation des fonctions d'une variable réelle, Paris, 1919, pp. 7–8.

† Cf., e. g., D. Jackson, On the approximate representation of an indefinite integral and the degree of convergence of related Fourier's series, Trans. Amer. Math. Soc., vol. 14 (1913), pp. 343–364; p. 350.

value of n, a trigonometric sum $t_n(x)$, of the nth order, such that, if ε_n is the maximum of $|F(x)-t_n(x)|$,

$$\lim_{n=\infty} \varepsilon_n \sqrt{n} = 0. \tag{5}$$

Let such a sequence of trigonometric sums be chosen, once for all, to be used throughout the proof. Since $c<F(x)<d$ for all values of x, and since $F(x)$ is continuous, the differences $F(x)-c$ and $d-F(x)$ will each have a positive minimum, and as soon as ε_n is less than the smaller of these minima, it will be certain that

$$c<t_n(x)<d.$$

There is no loss of generality in supposing these inequalities to hold from the beginning, for the sums corresponding to the earlier values of n, finite in number, can be chosen in any way that may be convenient.

Let L be the greater of the numbers $|c|$, $|d|$, so that

$$|t_n(x)|<L$$

for all values of n and x, and let further notation be introduced as follows:

$$\begin{aligned} F(x)-t_n(x) &= \varphi_n(x), & y-t_n(x) &= \eta, & z-t_n(x) &= \zeta, \\ w[x, y+t_n(x)] &= W_n(x, y), & c-L &= C, & d+L &= D. \end{aligned}$$

It is to be noticed that C and D are independent of n, and that $|\varphi_n(x)| \leqq \varepsilon_n$. Since $w(x, y)$ is identically zero except for $c<y<d$, it follows that $W_n(x, y)$ is zero if y is not between $c-t_n(x)$ and $d-t_n(x)$, and

$$C<c-t_n(x)<d-t_n(x)<D.$$

It is seen that

$$\int_c^d |z-y|^m w(x, y)\, dy = \int_{c-t_n(x)}^{d-t_n(x)} |\zeta-\eta|^m w[x, \eta+t_n(x)]\, d\eta$$

$$= \int_C^D |\zeta-\eta|^m W_n(x, \eta)\, d\eta = \int_C^D |\zeta-y|^m W_n(x, y)\, dy.$$

From the equality of the first and last integrals it follows that if z is determined so as to minimize the first integral, the corresponding value

of ζ will make the last integral a minimum; in other words, $F(x)$ being the mth power average for the weight-function $w(x, y)$, the corresponding average for $W_n(x,y)$ is $\varphi_n(x)$.

Similarly, if

$$T_n(x) - t_n(x) = \tau_n(x),$$

it is found that

$$\int_0^{2\pi} dx \int_c^d w(x, y) \, |y - T_n(x)|^m \, dy = \int_0^{2\pi} dx \int_C^D W_n(x, y) \, |y - \tau_n(x)|^m \, dy,$$

and when $T_n(x)$ is the sum which minimizes the left-hand integral, $\tau_n(x)$ does the same for the integral on the right. So the problem of investigating the order of magnitude of $F(x) - T_n(x)$ is referred to a corresponding problem in which the average $\varphi_n(x)$ is always small; the difference $F(x) - T_n(x)$ is the same as the difference $\varphi_n(x) - \tau_n(x)$. It will be understood henceforth that $T_n(x)$ and $\tau_n(x)$ are the sums which reduce the integrals to their smallest value. During the next stages of the work, n will be held fast; the relation (5), for n becoming infinite, will be recalled at the very end of the demonstration.

To proceed with the reduced problem, let z once more represent a variable taking on arbitrary values, and, for any given value of x, let z_0 stand for the particular quantity $\varphi_n(x)$. Let

$$G(z, x) = \int_C^D |z - y|^m \, W_n(x, y) \, dy;$$

a more complete notation would of course be $G_n(z, x)$, but the omission of the subscript will not cause any misunderstanding. Since it has been assumed that $m > 2$, $G(z, x)$ will have continuous first and second derivatives with regard to z, given respectively by the formulas

$$G'(z, x) = m \int_C^D |z - y|^{m-1} \operatorname{sgn}(z - y) \, W_n(x, y) \, dy,$$

$$G''(z, x) = m(m - 1) \int_C^D |z - y|^{m-2} \, W_n(x, y) \, dy,$$

the accents indicating differentiation with respect to z.

There will be occasion to make use of certain inequalities derived from the fact that

$$\int_C^D |z-y|^{m-2} W_n(x,y)\,dy = \int_{c-t_n(x)}^{d-t_n(x)} |z-y|^{m-2} W_n(x,y)\,dy$$

$$= \int_c^d |z+t_n(x)-y|^{m-2} w(x,y)\,dy,$$

$$G''(z,x) = m(m-1)\int_c^d |z+t_n(x)-y|^{m-2} w(x,y)\,dy.$$

Let

$$g(z,x) = \int_c^d |z-y|^{m-2} w(x,y)\,dy.$$

This is a continuous function of its arguments, and is always positive. It has the period 2π with regard to x, and becomes infinite, uniformly with regard to x, if z goes to infinity in either direction. Consequently it has a positive minimum; this minimum, for convenience in the subsequent work, will be denoted by $2v/[m(m-1)]$. On the other hand, if z is restricted to a finite interval, $c \leqq z \leqq d$, say, while x varies without restriction, $g(z,x)$ will have a finite maximum, which may be denoted by $2V/[m(m-1)]$. Then

$$G''(z,x) \leqq 2V$$

if $z+t_n(x)$ is in the interval (c, d), and

$$G''(z,x) \geqq 2v > 0$$

for all values of z and x. It is to be noticed that v and V are independent of n.

Since $G'(z_0, x) = 0$, because of the fact that z_0 reduces $G(z,x)$ to a minimum,

$$G(z,x) = G(z_0,x) + \tfrac{1}{2}(z-z_0)^2 G''(\xi, x), \tag{6}$$

where ξ is a value between z_0 and z. This relation will be used first to compare $G(0,x)$ with $G(z_0,x)$. It will be recalled that

$$c < t_n(x) < d,$$

and also that the quantity $F(x) = z_0 + t_n(x)$ is between the same limits. Consequently, since ξ is now between 0 and z_0, it is certain that $\xi + t_n(x)$ is between c and d, and

$$G''(\xi, x) \leqq 2V.$$

From this it follows that

$$G(0, x) \leqq G(z_0, x) + V z_0^2,$$

or, since

$$|z_0| = |\varphi_n(x)| = |F(x) - t_n(x)| \leqq \epsilon_n,$$

that

$$G(0, x) \leqq G(z_0, x) + V\epsilon_n^2. \tag{7}$$

On the other hand, let $z = \tau_n(x)$ in (6). It is certain that

$$G''(\xi, x) \geq 2v,$$

whether $\xi + t_n(x)$ is between c and d or not, and therefore

$$G[\tau_n(x), x] \geq G(z_0, x) + v[\tau_n(x) - z_0]^2. \tag{8}$$

Now let $\mu_n = |\tau_n(x_0)|$ be the maximum of $|\tau_n(x)|$, and let it be supposed temporarily that $\mu_n \geqq 4\epsilon_n$. If this were not the case, it would mean simply that the next few steps could be dispensed with. By Bernstein's theorem,*

$$|\tau_n'(x)| \leqq n\mu_n$$

for all values of x, and

$$|\tau_n(x) - \tau_n(x_0)| \leqq n\mu_n |x - x_0|.$$

If x differs from x_0 by not more that $1/(2n)$, it will follow that

$$|\tau_n(x) - \tau_n(x_0)| \leqq \mu_n/2, \qquad |\tau_n(x)| \geq \mu_n/2.$$

For such values of x, since $|z_0| \leqq \epsilon_n \leqq \mu_n/4$,

$$|\tau_n(x) - z_0| \geq \mu_n/4,$$

and, from (8),

$$G[\tau_n(x), x] \geq G(z_0, x) + v(\mu_n/4)^2. \tag{9}$$

* Cf. de la Vallée Poussin, op. cit., pp. 39-42.

Let

$$\gamma_n = \int_0^{2\pi} dx \int_C^D W_n(x, y)\,|y - \tau_n(x)|^m\, dy = \int_0^{2\pi} G[\tau_n(x), x]\, dx,$$

$$\gamma = \int_0^{2\pi} G(z_0, x)\, dx = \int_0^{2\pi} G[\varphi_n(x), x]\, dx.$$

Since $G[\tau_n(x), x] \geqq G(z_0, x)$ for all values of x, and since (9) holds throughout an interval of length $1/n$,

$$\gamma_n \geqq \gamma + \frac{v}{n}\left(\frac{\mu_n}{4}\right)^2. \tag{10}$$

But if

$$\gamma' = \int_0^{2\pi} G(0, x)\, dx$$

is the result of replacing $\tau_n(x)$ by 0 in γ_n, it must be that $\gamma' \geqq \gamma_n$, because 0 can be regarded as a trigonometric sum of order n, and $\tau_n(x)$ is by definition the sum which gives the integral its smallest value. Also, because of (7), $\gamma' \leqq \gamma + 2\pi V\varepsilon_n^2$, and therefore

$$\gamma_n \leqq \gamma + 2\pi V\varepsilon_n^2. \tag{11}$$

From (10) and (11),

$$\frac{v}{n}\left(\frac{\mu_n}{4}\right)^2 \leqq 2\pi V\varepsilon_n^2,$$

$$\mu_n \leqq \sqrt{\frac{2\pi V}{v}} \cdot 4\varepsilon_n \sqrt{n}. \tag{12}$$

The last relation, obtained on the hypothesis that $\mu_n \geqq 4\varepsilon_n$, clearly holds in the contrary case as well, since $\sqrt{2\pi Vn/v} > 1$.

The proof of convergence is now practically complete. As μ_n is the maximum of $|\tau_n(x)|$, the relation (12) means that

$$|\tau_n(x)| \leqq k\varepsilon_n \sqrt{n}$$

for all values of x, with a coefficient k which is independent of x and n. Since $|\varphi_n(x)| \leqq \varepsilon_n$, it follows further that

$$|\varphi_n(x) - \tau_n(x)| \leqq (k+1)\varepsilon_n \sqrt{n}.$$

But $\varphi_n(x) - \tau_n(x)$ is the same as $F(x) - T_n(x)$; and as $\lim_{n=\infty} \varepsilon_n \sqrt{n} = 0$, it is recognized that $T_n(x)$ converges uniformly to the limit $F(x)$ when n becomes infinite.

The result thus obtained is so far in agreement with the corresponding result of § 3 that both can be covered by a single statement. To summarize hypotheses and conclusions in the form of a theorem:

If $m \geqq 2$, *and if*

$$\lim_{\delta=0} \omega(\delta)/\sqrt{\delta} = 0,$$

where $\omega(\delta)$ *is the modulus of continuity of* $F(x)$ (*in particular, if* $w(x, y)$ *has a continuous derivative with regard to* x), *then*

$$\lim_{n=\infty} T_n(x) = F(x),$$

uniformly in x.

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AN IRREGULAR BOUNDARY VALUE AND EXPANSION PROBLEM.

By L. E. Ward.

1. **Introduction.** Birkhoff* has treated a general class of boundary value and expansion problems, the class which he called regular. The condition of regularity is one bearing on the boundary conditions of the differential system under consideration, and the restriction to this class was convenient in the convergence proof for the allied expansions. Hopkins† treated a boundary value and expansion problem which does not satisfy Birkhoff's condition of regularity, namely the problem for the system

$$\frac{d^3u}{dx^3} + \rho^3 u = 0 \qquad u(0) = u'(0) = u(\pi) = 0.$$

The present paper is an extension of Hopkins' work to the case of the problem for the system

$$(1) \qquad \frac{d^n u}{dx^n} + \rho^n u = 0 \qquad u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u(\pi) = 0,$$

and the methods used and results obtained are closely parallel to Hopkins'. It is found that the expansions allied with the above system of order n can converge only if the function expanded is analytic in a certain regular n-sided polygon, and that, if the function satisfies certain auxiliary conditions also, the expansion will converge to the function.

2. **Preliminaries.** Assume $u = e^{\omega\rho x}$ and determine ω so that u will be a solution of the differential equation (1). We see that ω must satisfy the algebraic equation $\omega^n + 1 = 0$. Hence, if $\omega = e^{\frac{\pi i}{n}}$, n linearly independent solutions of the differential equation (1) are $e^{\omega^{2k-1}\rho x}$, $(k = 1, 2, \cdots n)$. Hence the general solution of the differential equation (1) is

$$u = \sum_{k=1}^{n} A_k e^{\omega^{2k-1}\rho x}.$$

* G. D. Birkhoff, Trans. Am. Math. Soc., 1908, pp. 219–231 and pp. 373–395.

† J. W. Hopkins, published by D. Jackson in Trans. Am. Math. Soc., 1919, p. 245, et seq.

Imposing the first $n-1$ boundary conditions of (1) gives the equations

$$\sum_{k=1}^{n} A_k = 0, \qquad \sum_{k=1}^{n} \omega^{2(k-1)} A_k = 0, \cdots \sum_{k=1}^{n} \omega^{2(k-1)(n-2)} A_k = 0.$$

These equations determine the ratios of the A's, giving

$$A_1 : A_2 : \cdots : A_n = \omega : \omega^3 : \cdots : \omega^{2n-1}.$$

Hence, at least to a multiplying constant, the solution is

$$u = \omega e^{\omega\rho x} + \omega^3 e^{\omega^3\rho x} + \cdots + \omega^{2n-1} e^{\omega^{2n-1}\rho x}.$$

We shall denote this function of x and ϱ by $\delta(\varrho x)$.

Use of the last boundary condition of (1) gives the characteristic equation, namely,

$$\delta(\varrho\pi) = \omega e^{\omega\rho\pi} + \omega^3 e^{\omega^3\rho\pi} + \cdots + \omega^{2n-1} e^{\omega^{2n-1}\rho\pi} = 0.$$

If we write $\varphi(\varrho) = 1 + \omega^{2(n-1)} e^{(\omega^{2n-1}-\omega)\rho\pi}$ and

$$\psi(\varrho) = \omega^2 e^{(\omega^3-\omega)\rho\pi} + \omega^4 e^{(\omega^5-\omega)\rho\pi} + \cdots + \omega^{2(n-2)} e^{(\omega^{2n-3}-\omega)\rho\pi},$$

so that $\delta(\varrho\pi) = \omega e^{\omega\rho\pi}[\varphi(\varrho) + \psi(\varrho)]$, we can write the characteristic equation as $\varphi(\varrho) + \psi(\varrho) = 0$.

If $\bar{\lambda}$ means the conjugate of λ, we can see that $\overline{\delta(\varrho\pi)} = \delta(\bar{\varrho}\pi)$ merely by writing out both sides of this equation with $\varrho = \xi + \eta i$ and rearranging terms. For this reason we can, in considering the characteristic equation, restrict ourselves to the sector $-\frac{\pi}{n} \leqq \arg\varrho \leqq 0$ instead of working in the entire sector $-\frac{\pi}{n} \leqq \arg\varrho \leqq \frac{\pi}{n}$.

Let $\varrho = |\varrho| e^{i\theta}$; then if R means "the real part of", we have

$$R[(\omega^j - \omega)\varrho] = |\varrho| \left[\cos\theta\left(\cos\frac{\pi j}{n} - \cos\frac{\pi}{n}\right) - \sin\theta\left(\sin\frac{\pi j}{n} - \sin\frac{\pi}{n}\right)\right]$$

$$= -2|\varrho| \sin\left[\theta + \frac{\pi(j+1)}{2n}\right] \sin\frac{\pi(j-1)}{2n}.$$

We now restrict ourselves to the sector $-\frac{\pi}{n} \leqq \theta \leqq 0$. Suppose $3 \leqq j \leqq 2n-3$, which is the case for every term of $\psi(\varrho)$. Then

$\frac{\pi}{n} \leqq \frac{\pi(j-1)}{2n} \leqq \pi - \frac{2\pi}{n}$, so that $\sin\frac{\pi(j-1)}{2n} \geqq \sin\frac{\pi}{n}$. Also $\frac{\pi}{n} \leqq \theta + \frac{\pi(j+1)}{2n} \leqq \pi - \frac{\pi}{n}$, so that $\sin\left[\theta + \frac{\pi(j+1)}{2n}\right] \geqq \sin\frac{\pi}{n}$. Hence $R[(\omega^j - \omega)\varrho] \leqq -2|\varrho|\sin^2\frac{\pi}{n}$. Hence we can make $|\psi(\varrho)|$ as small as we like in this sector by taking $|\varrho|$ large enough.

Hence, in our chosen sector, an approximation to $\delta(\varrho\pi) = 0$ is $\varphi(\varrho) = 0$. Solving for ϱ, and calling ϱ_k' the solution, we obtain $\varrho_k' = \left(2k+1-\frac{2}{n}\right)\Big/2\sin\frac{\pi}{n}$, k any integer. It is clear that these numbers are simple roots of $\varphi(\varrho) = 0$.

Let us cut from the ϱ-plane small circles of radius r and centers ϱ_k'. We denote the remaining part of the sector $-\frac{\pi}{n} \leqq \theta \leqq 0$ by S^-, that of the sector $0 \leqq \theta \leqq \frac{\pi}{n}$ by S^+. If we write $\varrho = \xi + \eta i$, we have $\varphi(\varrho) = 1 + \omega^{2(n-1)} e^{2\pi\eta\sin\frac{\pi}{n}} e^{-2\pi i\xi\sin\frac{\pi}{n}}$, which shows that, as $\eta \to -\infty$, $\varphi(\varrho) \to 1$ uniformly. Also $\varphi(\varrho)$ has $1/\sin\frac{\pi}{n}$ for its period. Hence in S^- we can write $|\varphi(\varrho)| \geqq \delta$, where δ is a positive constant.

Choose a number R such that if $|\varrho| \geqq R$ and ϱ is in S^-, then $|\psi(\varrho)| \leqq \delta/2$. In this region we shall have

$$|\delta(\varrho\pi)| = |e^{\omega\rho\pi}|\cdot|\varphi(\varrho) + \psi(\varrho)| \geqq |e^{\omega\rho\pi}|[|\varphi(\varrho)| - |\psi(\varrho)|] \geqq \frac{\delta}{2}|e^{\omega\rho\pi}|.$$

Hence if $|\varrho| \geqq R$ and ϱ is in S^-, $\delta(\varrho\pi)$ cannot vanish.

If $|\varrho| \geqq R$ and ϱ is in S^-, then $|\overline{\varrho}| \geqq R$ and $\overline{\varrho}$ is in S^+. If $\delta(\overline{\varrho}\pi)$ vanishes under these conditions, then so does $\overline{\delta(\varrho\pi)}$, and hence $\delta(\varrho\pi)$. This contradicts what we have shown above. Hence if ϱ is in $S = S^- + S^+$ and $|\varrho| \geqq R$, then $\delta(\varrho\pi)$ cannot vanish.

Let C_k denote the small circle about the point ϱ_k'. As ϱ passes once around C_k, $\arg\varphi(\varrho)$ increases by 2π, i. e., the point $\varphi(\varrho)$ travels once around the origin in the φ-plane, and its minimum distance from that origin is not less than δ. If $|\varrho| \geqq R$, we have $|\psi(\varrho)| < \delta/2$, and so the point $\varphi(\varrho) + \psi(\varrho)$ must at the same time travel once around the origin of its plane. Hence $\delta(\varrho\pi)$ has just one zero in C_k.

From this we conclude that $\delta(\varrho\pi)$ has an infinitude of zeros in the sector $-\frac{\pi}{n} \leqq \theta \leqq \frac{\pi}{n}$ for $|\varrho| \geqq R$, which are all given by $\varrho_k = \left(2k+1-\frac{2}{n}\right)\Big/2\sin\frac{\pi}{n} + \varepsilon_k$, where $\varepsilon_k \to 0$ as $k \to \infty$. We can conclude further that ε_k is real. For, if ε_k is not real, we have, for the

corresponding ϱ_k, $\overline{\delta(\varrho_k \pi)} = 0$. Hence $\delta(\bar{\varrho}_k \pi) = 0$. Hence there are two zeros ϱ_k and $\bar{\varrho}_k$ of $\delta(\varrho \pi)$ inside C_k. This we know to be untrue. Hence ϱ_k is real if $|\varrho| \geqq R$.

Also, $u_k(x) \equiv \delta(\varrho_k x)$ is real if ϱ_k is real, x being a real variable. We have

$$\begin{aligned}
\delta(\varrho_k x) &= \omega e^{\omega \rho_k x} + \omega^3 e^{\omega^3 \rho_k x} + \cdots + \omega^{2n-1} e^{\omega^{2n-1} \rho_k x} \\
&= e^{\rho_k x \cos \frac{\pi}{n}} \left[\cos\left(\frac{\pi}{n} + \varrho_k x \sin \frac{\pi}{n}\right) + i \sin\left(\frac{\pi}{n} + \varrho_k x \sin \frac{\pi}{n}\right) \right] \\
&\quad + e^{\rho_k x \cos \frac{3\pi}{n}} \left[\cos\left(\frac{3\pi}{n} + \varrho_k x \sin \frac{3\pi}{n}\right) + i \sin\left(\frac{3\pi}{n} + \varrho_k x \sin \frac{3\pi}{n}\right) \right] \\
&\quad + - \\
&\quad + e^{\rho_k x \cos \frac{(2n-1)\pi}{n}} \left[\cos\left\{\frac{(2n-1)\pi}{n} + \varrho_k x \sin \frac{(2n-1)\pi}{n}\right\} \right. \\
&\quad \left. + i \sin\left\{\frac{(2n-1)\pi}{n} + \varrho_k x \sin \frac{(2n-1)\pi}{n}\right\} \right].
\end{aligned}$$

From this we see that the imaginary term in the first line cancels that in the last; the imaginary term in the second line that in the next to the last, and so on. If the number of lines is even, all imaginary terms cancel in this way. If it is odd, there is no term to cancel the term $i e^{\rho_k x \cos \frac{n\pi}{n}} \sin\left(\frac{n\pi}{n} + \varrho_k x \sin \frac{n\pi}{n}\right)$, but this term is zero. Hence

$$\delta(\varrho_k x) = \sum_{l=1}^{n} e^{\rho_k x \cos \frac{(2l-1)\pi}{n}} \cos\left[\frac{(2l-1)\pi}{n} + \varrho_k x \sin \frac{(2l-1)\pi}{n}\right],$$

from which expression the reality of $\delta(\varrho_k x)$ is clear.

3. **Adjoint Relations.** If we set $L(u) \equiv u^{(n)} + \varrho^n u$, the given differential system becomes

$$L(u) = 0, \quad u(0) = u'(0) = \cdots = u^{(n-2)}(0) = u(\pi) = 0.$$

The adjoint system is

$$M(v) = 0, \quad v(\pi) = v'(\pi) = \cdots = v^{(n-2)}(\pi) = v(0) = 0,$$

where $M(v) \equiv (-1)^n v^{(n)} + \varrho^n v$. The general solution of $M(v) = 0$ is

$$v = A_1 e^{-\omega \rho x} + A_2 e^{-\omega^3 \rho x} + \cdots + A_n e^{-\omega^{2n-1} \rho x},$$

and the first $n-1$ adjoint boundary conditions give

$$A_1 : A_2 : \cdots : A_n = \omega\, e^{\omega\rho\pi} : \omega^3\, e^{\omega^3\rho\pi} : \cdots : \omega^{2n-1}\, e^{\omega^{2n-1}\rho\pi}.$$

Hence, to a multiplying constant, we have $v(x) = \delta[\rho(\pi - x)]$. The characteristic equation for the adjoint system, $\delta(\rho\pi) = 0$, is the same as that for the given system.

If $u_k(x)$ and $v_l(x)$ are solutions of the given system and its adjoint system and correspond to ρ_k and ρ_l respectively, we have

$$\int_0^\pi (v_l u_k^{(n)} - u_k v_l^{(n)})\, dx = \int_0^\pi (-\rho_k^n u_k v_l + \rho_l^n u_k v_l)\, dx = (\rho_l^n - \rho_k^n) \int_0^\pi u_k v_l\, dx.$$

But by Green's formula* and the boundary conditions

$$\int_0^\pi (v_l u_k^{(n)} - u_k v_l^{(n)})\, dx = (-1)^{n-1} \left[u_k v_l^{(n-1)} - u_k' v_l^{(n-2)} + \cdots + (-1)^{n-1} u_k^{(n-1)} v_l\right]_0^\pi = 0.$$

Also, if $k \neq l$, then $\rho_l \neq \rho_k$, so that $\int_0^\pi u_k(x)\, v_l(x)\, dx = 0$ $(k \neq l)$.

Further, direct computation gives

$$\int_0^\pi u_j(x)\, v_j(x)\, dx = \sum_{k,l=1}^{n} \omega^{2(k+l-1)}\, e^{\omega^{2l-1}\rho_j\pi} \int_0^\pi e^{(\omega^{2k-1} - \omega^{2l-1})\rho_j x}\, dx$$

$$= \pi \sum_{k=1}^{n} \omega^{2(2k-1)}\, e^{\omega^{2k-1}\rho_j\pi} + \frac{2}{\rho_j} \sum_{k,l=1}^{n}{}^{*} \frac{\omega^{2l-1}}{1 - \omega^{2(l-k)}} \left(e^{\omega^{2k-1}\rho_j\pi} - e^{\omega^{2l-1}\rho_j\pi}\right),$$

where in the latter summation all terms for which $k = l$ are to be omitted. Use of the characteristic equation shows that this summation is zero. Hence

$$\int_0^\pi u_j\, v_j\, dx = \pi \sum_{k=1}^{n} \omega^{2(k-1)}\, e^{\omega^{2k-1}\rho_j\pi}.$$

These relations allow us to determine the coefficients a_k in the tentative equation $f(x) = \sum_1^\infty a_k u_k(x)$ in the usual way. When the coefficients are thus obtained, we shall call the resulting series the formal expansion of $f(x)$.

* Cf. M. Bôcher, Leçons sur les Méthodes de Sturm, p. 30.

4. The necessary conditions. We derive in this section conditions which a function $f(x)$ must satisfy in order that its formal expansion may converge to $f(x)$.

Suppose we have a series $\sum_1^\infty a_k u_k(x)$, where the a's are constants, which converges uniformly in an interval I: $x_1 \leqq x \leqq x_0$, where $0 \leqq x_1 < x_0$. Then there exists a constant g such that $|a_k u_k(x)| \leqq g$ throughout I and for all k. Now we have

$$|u_k(x)| = |e^{\omega\rho_k x}| \cdot |1 + \omega^2 e^{(\omega^2-\omega)\rho_k x} + \cdots + \omega^{2(n-1)} e^{(\omega^{2n-1}-\omega)\rho_k x}|$$

$$\geqq |e^{\omega\rho_k x}| \cdot \left[|1+\omega^{2(n-1)} e^{(\omega^{2n-1}-\omega)\rho_k x}| - |\omega^2 e^{(\omega^3-\omega)\rho_k x} + \cdots + \omega^{2(n-2)} e^{(\omega^{2n-2}-\omega)\rho_k x}|\right].$$

By taking k large enough, there will be a number x_k' in I such that $\rho_k x_k' \sin\frac{\pi}{n} = m\pi$, where m is a positive integer. Hence $e^{(\omega^{2n-1}-\omega)\rho_k x_k'} = 1$, so that

$$|1+\omega^{2(n-1)} e^{(\omega^{2n-1}-\omega)\rho_k x_k'}| = |1+\omega^2| = \sqrt{2+2\cos\frac{2\pi}{n}} \geqq 3/2, \quad (n \geqq 3).$$

Also, if k is large enough, we have

$$|\omega^2 e^{(\omega^3-\omega)\rho_k x_k'} + \cdots + \omega^{2(n-2)} e^{(\omega^{2n-3}-\omega)\rho_k x_k'}| < 1.$$

Hence

$$|u_k(x_k')| > |e^{\omega\rho_k x_k'}|(\sqrt{3/2}-1) = (\sqrt{3/2}-1)e^{\rho_k x_k' \cos\frac{\pi}{n}},$$

and, further,

$$|a_k| \leqq \frac{g}{|u_k(x_k')|} \leqq \frac{g}{\sqrt{3/2}-1} e^{-\rho_k x_k' \cos\frac{\pi}{n}} \leqq h e^{-\rho_k x_1 \cos\frac{\pi}{n}} \qquad (k \geqq K).$$

This result can now be extended to hold for smaller values of k than K by taking a different positive constant h. Thus we have

THEOREM 1: *If the series $\sum_1^\infty a_k u_k(x)$ converges uniformly in any interval reaching out to $x_0 > 0$, and if x_1 is any number less than x_0, then there will be a positive number h independent of x and such that $|a_k| < h e^{-\rho_k x_1 \cos\frac{\pi}{n}}$ for all k.*

We proceed to investigate the convergence of $\sum_1^\infty a_k u_k(x)$ for complex values of x, under the hypotheses of this theorem. Let $x = \xi + \eta i$, and choose a number x_2 such that $0 < x_2 < x_1$. Now

$$|\omega e^{\omega\rho_k x}| = e^{\rho_k \xi \cos\frac{\pi}{n} - \rho_k \eta \sin\frac{\pi}{n}} \leqq e^{\rho_k x_2 \cos\frac{\pi}{n}}$$

provided $\xi\cos\frac{\pi}{n} - \eta\sin\frac{\pi}{n} \leqq x_2 \cos\frac{\pi}{n}$;

$$|\omega^3 e^{\omega^3 \rho_k x}| = e^{\rho_k \xi \cos\frac{3\pi}{n} - \rho_k \eta \sin\frac{3\pi}{n}} \leqq e^{\rho_k x_2 \cos\frac{\pi}{n}}$$

provided $\xi \cos\dfrac{3\pi}{n} - \eta \sin\dfrac{3\pi}{n} \leqq x_2 \cos\dfrac{\pi}{n}$; and so on to

$$|\omega^{2n-1} e^{\omega^{2n-1}\rho_k x}| = e^{\rho_k \xi \cos\frac{(2n-1)\pi}{n} - \rho_k \eta \sin\frac{(2n-1)\pi}{n}} \leqq e^{\rho_k x \cos\frac{\pi}{n}}$$

provided $\xi \cos\dfrac{(2n-1)\pi}{n} - \eta \sin\dfrac{(2n-1)\pi}{n} \leqq x_2 \cos\dfrac{\pi}{n}$. The lines

$$\xi \cos\frac{(2k-1)\pi}{n} - \eta \sin\frac{(2k-1)\pi}{n} = x_2 \cos\frac{\pi}{n}, \quad (k = 1, 2, \cdots n)$$

form a regular polygon of n sides, center $x = 0$, vertices x_2, $\omega^2 x_2$, $\omega^4 x_2$, etc. If x is inside this polygon or on its boundary, all of the above inequalities are satisfied. Hence we have in this case $|u_k(x)| \leqq n e^{\rho_k x_2 \cos\frac{\pi}{n}}$, and therefore $|a_k u_k(x)| < h n l^{(x_2 - x_1)\rho_k \cos\frac{\pi}{n}}$. But this is the general term of a convergent series of positive constants. Hence the series $\sum_1^\infty a_k u_k(x)$ converges uniformly in the interior and on the boundary of the polygon. Since each term of the series is analytic in x, we have

THEOREM 2: *Under the hypotheses of Theorem* 1, *the series* $\sum_1^\infty a_k u_k(x)$ *converges uniformly, and represents a function analytic in the interior of a regular n-sided polygon whose center is* $x = 0$ *and one vertex of which is at* x_0.

If X is the upper limit of the possible x_0's, the polygon similar to the one of Theorem 2, similarly placed, with the same center, but having one vertex at X, is the actual region of convergence of the series, except possibly for points on the rays $\arg x = \dfrac{2k\pi}{n}$, $(k = 1, 2, \cdots n)$. For, suppose the series converges at $x = x_2'$, where x_2' is outside of the larger polygon and $\arg x_2' \neq \dfrac{2k\pi}{n}$, $(k = 1, 2, \cdots n-1)$. Let us expand the polygon keeping it similar to its original form, similarly placed, and center $x = 0$, until one side passes through x_2'; and call X' the vertex of the expanded polygon which is real and positive. We shall see that the series converges uniformly for $\arg x = \arg x_2'$ and $|x| \leqq |x_2'|$. Calling $\theta = \arg x_2$, we write $x = r e^{i\theta}$ and $x_2' = r_2' e^{i\theta}$, where $r = |x|$ and $r_2' = |x_2'|$. Consider first the case in which x_2' is located so that $0 < \theta < \dfrac{2\pi}{n}$,

and write

$$\frac{u_k(x)}{u_k(x_2')} = \frac{\sum_{l=1}^{n} \omega^{2l-1}\, e^{\omega^{2l-1}\rho_k x}}{\sum_{l=1}^{n} \omega^{2l-1}\, e^{\omega^{2l-1}\rho_k x_2'}} = \frac{\sum_{l=1}^{n} \omega^{2l-1}\, e^{\rho_k(\omega^{2l-1}x - \omega^{2n-1}x_2')}}{\sum_{l=1}^{n} \omega^{2l-1}\, e^{\rho_k x_2'(\omega^{2l-1} - \omega^{2n-1})}}.$$

We have

$$R[\omega^{2l-1}x - \omega^{2n-1}x_2] = r\cos\left[\frac{(2k-1)\pi}{n} + \theta\right] - r_2'\cos\left[\frac{(2n-1)\pi}{n} + \theta\right],$$

whence $R[\omega^{2l-1}x - \omega^{2n-1}x_2'] < 0$ for $l = 1, 2, \cdots n$, since $r \leqq r_2'$. Also

$$R[x_2'(\omega^{2l-1} - \omega^{2n-1})] = r_2'\left[\cos\left\{\frac{(2l-1)\pi}{n} + \theta\right\} - \cos\left\{\frac{(2n-1)\pi}{n} + \theta\right\}\right],$$

whence $R[x_2'(\omega^{2l-1} - \omega^{2n-1})] < 0$ for $l = 1, 2, \cdots n-1$.

Also, the last term in the denominator of the last fraction for $u_k(x)/u_k(x_2')$ above equals ω^{2n-1}. Hence $|u_k(x)/u_k(x_2')| \to 0$ as $k \to \infty$, so that the series does converge on the part of the ray $\arg x = \theta$ which is interior to the enlarged polygon. A similar result is obtained in the same way if $\frac{2k\pi}{n} < \theta < \frac{2(k+1)\pi}{n}$.

Now one term of $u_k(x)$ will have at x_2' a numerical value greater than the numerical value of any other term of $u_k(x)$ at x_2'. Suppose this term is $\omega^{2l-1}\, e^{\omega^{2l-1}\rho_k x}$; its numerical value at x_2' is $e^{\rho_k X'\cos\frac{\pi}{n}}$, and it has this same numerical value all along the side of the polygon passing through x_2'. We call this term the preponderant term of $u_k(x)$ for this side of the polygon. Along each other side of the polygon there will be a preponderant term, and its numerical value will be in every case $e^{\rho_k X'\cos\frac{\pi}{n}}$.

We can now show the series to be convergent when $x = X'$. We have

$$|u_k(x_2')| > |\omega^{2l-1}\, e^{\omega^{2l-1}\rho_k x_2}| - \left|\sum_{j=1}^{n}{}^{*}\, \omega^{2j-1}\, e^{\omega^{2j-1}\rho_k x_2'}\right|,$$

the term in the summation for which $j = l$ being ommitted. Also

$$|u_k(X')| < \sum_{j=1}^{n} |e^{\omega^{2j-1}\rho_k X'}| < n\, e^{\rho_k X'\cos\frac{\pi}{n}}.$$

Hence

$$\left|\frac{u_k(X')}{u_k(x_2')}\right| < \frac{n e^{\rho_k X' \cos\frac{\pi}{n}}}{e^{\rho_k X' \cos\frac{\pi}{n}} - \left|\sum_{j=1}^{n}{}^{*}\, \omega^{2j-1}\, e^{\omega^{2j-1}\rho_k x_2'}\right|}$$

$$= \frac{n}{1 - e^{-\rho_k X' \cos\frac{\pi}{n}}\left|\sum_{j=1}^{n}{}^{*}\, \omega^{2j-1}\, e^{\omega^{2j-1}\rho_k x_2'}\right|},$$

which tends toward n as a limit as $k \to \infty$. Hence the series converges for $x = X'$.

The argument in the above paragraph can be carried through in the same way if x_2' is any point on the ray $\arg x = \theta$ for which the series converges. Hence the series converges uniformly in the interval $(0, x_0')$, where $0 < x_0' < X'$. But this is contrary to hypothesis. Hence the polygon whose vertex to the extreme right is X is the actual region of convergence of the series except possibly for points on the rays $\arg x = 2k\pi/n$, $(k = 1, 2, \cdots n)$.

Denote by $f(x)$ the function represented by the series $\sum_1^\infty a_k u_k(x)$. It is clear that $f(x)$ is analytic in a certain n-sided regular polygon whose center is $x = 0$. Also the qth derivative of $f(x)$ is equal to the series obtained by differentiating term by term q times the series for $f(x)$. Hence, $u_k(0)$, $u_k'(0)$, $\cdots$ $u^{(n-2)}(0)$ each being zero for all k, we see that $f(0) = f'(0) = \cdots = f^{(n-2)}(0) = 0$. Hence $f(x) = x^{n-1}\varphi(x)$, and $\varphi(x)$ is analytic where $f(x)$ is. By direct computation we get $u_k(\omega^{2m}x) = \omega^{2(n-m)}u_k(x)$ for $m = 1, 2, \cdots n-1$. Hence $f(\omega^{2m}x) = \omega^{2(n-m)}f(x)$, and we have

$$\varphi(\omega^{2m}x) = f(\omega^{2m}x)/(\omega^{2m}x)^{n-1} = \omega^{2(n-m)}f(x)/\omega^{2mn-2m}x^{n-1} = \varphi(x).$$

Now, since $\varphi(x)$ is analytic at $x = 0$, we can write $\varphi(x) = \sum_0^\infty c_l x^l$. But also, from the above, $\varphi(x) = \frac{1}{n}\sum_1^n \varphi[\omega^{2(k-1)}x] = \frac{1}{n}\sum_0^\infty c_l\, \delta_l\, x^l$, where $\delta_l = 1 + \omega^{2l} + \omega^{4l} + \cdots + \omega^{2(n-1)l}$. Comparing the two series for $\varphi(x)$ we get $c_l = c_l\,\delta_l/n$, whence $c_l = 0$ unless $\delta_l = n$, which occurs when and only when $l = mn$, where m is a positive integer or zero. Hence $\varphi(x) = c_0 + c_n x^n + c_{2n}x^{2n} + \cdots$. We have, therefore, the following theorem.

THEOREM 3: *A necessary condition that $f(x)$ be capable of being represented by the series $\sum_1^\infty a_k u_k(x)$ is that $f(x) = x^{n-1}\psi(x^n)$, where $\psi(x^n)$ is a convergent power series in x^n.*

5. The Sufficient Conditions. It is known that the sum of the first m terms of the formal expansion of $f(x)$ is given by

$$I_m(x) = \frac{1}{2\pi i}\int_{\gamma_m}\int_0^{\pi} n\varrho^{n-1}\, G(x, s, \varrho)\, f(s)\, ds\, d\varrho,$$

where $G(x, s, \varrho)$ is the Green's function for our differential system, and γ_m is a circular arc of radius R_m, $\varrho_m < R_m < \varrho_{m+1}$, extending from $R_m\,\omega^{2n-1}$ to $R_m\,\omega$.* Using the formulae given by Birkhoff,† we find

$$G(x, s, \varrho) = \frac{1}{n\varrho^{n-1}\,\delta(\varrho\pi)}\{\delta(\varrho x)\,\delta[\varrho(\pi - s)] - \delta(\varrho\pi)\,\delta[\varrho(x - s)]\} \qquad (x > s),$$

$$G(x, s, \varrho) = \frac{\delta(\varrho x)\,\delta[\varrho(\pi - s)]}{n\varrho^{n-1}\,\delta(\varrho\pi)}. \qquad (x < s).$$

The main theorem of this paper is the following.

THEOREM 4: *If $f(x)$ is analytic in the interior and on the boundary of a circle, center $x = 0$, radius x_0, $0 < x_0 < \pi$, which includes in its power series expansion only powers of indices congruent to $n - 1$ (mod n), and which has a continuous second derivative for real values of x in $0 \leqq x \leqq \pi$, then the formal development of $f(x)$ converges uniformly to $f(x)$ in the interval $0 \leqq x \leqq x_0$.*

We shall show that $I_m(x)$ converges uniformly to $f(x)$ for $0 \leqq x \leqq x_0$. We have

$$\int_0^{\pi} n\varrho^{n-1}\, Gf\, ds = \left[\int_0^x + \int_x^{\pi}\right] n\varrho^{n-1}\, Gf\, ds$$

$$= \frac{\delta(\varrho x)}{\delta(\varrho\pi)}\int_0^{\pi}\delta[\varrho(\pi - s)]f(s)\,ds - \int_0^x \delta[\varrho(x - s)]\, f(s)\, ds.$$

But

$$\int_0^x \delta[\varrho(x - s)]\, f(s)\, ds = \sum_{k=1}^{n}\omega^{2k-1}\, e^{\omega^{2k-1}\rho x}\int_0^x e^{-\omega^{2k-1}\rho s}\, f(s)\, ds.$$

Integrating twice by parts, we find

$$\int_0^x e^{-\omega^{2k-1}\rho s}\, f(s)\, ds$$

$$= \frac{e^{-\omega^{2k-1}\rho x} f(x)}{-\omega^{2k-1}\varrho} - \frac{e^{-\omega^{2k-1}\rho x} f'(x)}{\omega^{2(2k-1)}\varrho^2} + \frac{1}{\omega^{2(2k-1)}\varrho^2}\int_0^x e^{-\omega^{2k-1}\rho s}\, f''(s)\, ds.$$

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* See Birkhoff, loc. cit. p. 379 and p. 390.

† Loc. cit. p. 378.

Hence

$$\int_0^x \delta[\varrho(x-s)]f(s)\,ds = -\frac{nf(x)}{\varrho} + \frac{1}{\varrho^2}\int_0^x \delta_1[\varrho(x-s)]f''(s)\,ds,$$

where $\delta_1[\varrho(x-s)] = \sum_{k=1}^{n} \omega^{2(n-k)+1} e^{\omega^{2k-1}\rho(x-s)}$. The sum of terms containing $f'(x)$ has $\sum_1^n \frac{1}{\omega^{2k-1}} = \sum_1^n \omega^{2k-1} = 0$ as a factor, and so does not appear. Replacing x by π, we get

$$\int_0^\pi \delta[\varrho(\pi-s)]f(s)\,ds = -\frac{nf(\pi)}{\varrho} + \frac{1}{\varrho^2}\int_0^\pi \delta_1[\varrho(\pi-s)]f''(s)\,ds.$$

It is because we use this formula that we require $f(x)$ to have a continuous second derivative in the interval $(0, \pi)$. Combining these two results, we get

$$\int_0^\pi n\varrho^{n-1}Gf\,ds = \frac{n}{\varrho}f(x) - nf(\pi)\frac{\delta(\varrho x)}{\varrho\,\delta(\varrho\pi)} + \frac{A}{\varrho^2\delta(\varrho\pi)},$$

where

$$A = \delta(\varrho x)\int_0^\pi \delta_1[\varrho(\pi-s)]f''(s)\,ds - \delta(\varrho\pi)\int_0^x \delta_1[\varrho(x-s)]f''(s)\,ds.$$

Now

$$\int_{\gamma_m} \frac{nf(x)}{\varrho}\,d\varrho = nf(x)\log\varrho\,\Big|_{\gamma_m} = nf(x)\log\omega^2 = 2\pi i f(x).$$

Hence

$$I_m(x) = f(x) - \frac{nf(\pi)}{2\pi i}\int_{\gamma_m}\frac{\delta(\varrho x)}{\varrho\,\delta(\varrho\pi)}\,d\varrho + \frac{1}{2\pi i}\int_{\gamma_m}\frac{A}{\varrho^2\delta(\varrho\pi)}\,d\varrho.$$

We shall now show that $\int_{\gamma_m}\frac{\delta(\varrho x)}{\varrho\,\delta(\varrho\pi)}\,d\varrho$ tends uniformly to zero as a limit when $m\to\infty$. Let us call γ_m^+ the part of γ_m in S^+, and γ_m^- the part of γ_m in S^-. Consider first $\int_{\gamma_m^-}\frac{\delta(\varrho x)}{\varrho\,\delta(\varrho\pi)}\,d\varrho$. If $|\varrho| \geqq R$ and ϱ is in S^-, we have $|\delta(\varrho\pi)| \geqq \frac{\delta}{2}|e^{\omega\rho\pi}|$. Also

$$|\delta(\varrho x)| \leqq \sum_{k=1}^{n}|e^{\omega^{2k-1}\rho x}| = \sum_{k=1}^{n} e^{\left[\xi\cos\frac{(2k-1)\pi}{n} - \eta\cos\frac{(2k-1)\pi}{n}\right]x},$$

where $\varrho = \xi + \eta i$. We shall show that the bracket in this exponent, regarded as a function of k, has its largest value when $k = 1$. We prove

first that $2\xi \sin\frac{k\pi}{n} \geqq -2\eta\cos\frac{k\pi}{n}$ for $k = 1, 2, \cdots n$ when ϱ satisfies $-\frac{\pi}{n} \leqq \arg\varrho \leqq 0$. When $k = n$, this inequality is obviously true, for then it reduces to $0 \geqq \eta$. When $\frac{\pi}{2} < \frac{k\pi}{n} < \pi$, we have $\xi \sin\frac{k\pi}{n} > 0$ and $-\eta\cos\frac{k\pi}{n} \leqq 0$, so that the inequality is true. It is obviously true when, if ever, $\frac{k\pi}{n} = \frac{\pi}{2}$. Finally, it is true when $0 < \frac{k\pi}{n} < \frac{\pi}{2}$, for then we can write it $\tan\frac{k\pi}{n} \geqq -\eta/\xi$. But $-\frac{\pi}{n} \leqq \arg\varrho \leqq 0$, so that $-\tan\frac{\pi}{n} \leqq \tan\arg\varrho \leqq 0$. Hence $-\tan\arg\varrho \leqq \tan\frac{\pi}{n}$. Also $\tan\frac{\pi}{n} \leqq \tan\frac{k\pi}{n}$. Hence $-\tan\arg\varrho = -\frac{\eta}{\xi} \leqq \tan\frac{k\pi}{n}$. We have now in succession the following inequalities

$$2\xi\sin\frac{k\pi}{n} \geqq -2\eta\cos\frac{k\pi}{n},$$

$$2\xi\sin\frac{k\pi}{n}\sin\frac{(k-1)\pi}{n} \geqq -2\eta\sin\frac{(k-1)\pi}{n}\cos\frac{k\pi}{n},$$

$$\xi\left[\cos\frac{\pi}{n} - \cos\frac{(2k-1)\pi}{n}\right] \geqq \eta\left[\sin\frac{\pi}{n} - \sin\frac{(2k-1)\pi}{n}\right],$$

$$\xi\cos\frac{\pi}{n} - \eta\sin\frac{\pi}{n} \geqq \xi\cos\frac{(2k-1)\pi}{n} - \eta\sin\frac{(2k-1)\pi}{n}.$$

These hold for $k = 1, 2, \cdots, n$, and in the sector $-\frac{\pi}{n} \leqq \arg\varrho \leqq 0$. Hence $|e^{\omega\rho x}| \geqq |e^{\omega^{2k-1}\rho x}|$ for $k = 1, 2, \cdots n$, so that $|\delta(\varrho x)| \leqq n|e^{\omega\rho x}|$. Hence on γ_m^- we have, if m is large,

$$\left|\frac{\delta(\varrho x)}{\delta(\varrho\pi)}\right| \leqq \frac{2n}{\delta}|e^{\omega\rho(x-\pi)}| = \frac{2n}{\delta}e^{\left(\xi\cos\frac{\pi}{n} - \eta\sin\frac{\pi}{n}\right)(x-\pi)},$$

or

$$\left|\frac{\delta(\varrho x)}{\delta(\varrho\pi)}\right| \leqq \frac{2n}{\delta}e^{\left(\xi\cos\frac{\pi}{n} - \eta\sin\frac{\pi}{n}\right)(x_0-\pi)} = \frac{2n}{\delta}|e^{\omega\rho(x_0-\pi)}|,$$

where $0 \leqq x \leqq x_0 < \pi$.

If in $\left|\int_{\gamma_m^-} \frac{\delta(\varrho x)}{\varrho\delta(\varrho\pi)}\,d\varrho\right| \leqq \int_{\gamma_m^-}\left|\frac{\delta(\varrho x)}{\varrho\delta(\varrho\pi)}\right||d\varrho|$, we write $\varrho = R_m e^{i\theta}$, we have

$$\left|\int_{\gamma_m^-} \frac{\delta(\varrho x)}{\varrho\delta(\varrho\pi)}\,d\varrho\right| \leqq \frac{2n}{\delta}\int_{-\frac{\pi}{n}}^{0} |e^{\omega\rho(x_0-\pi)}|\,d\theta$$

$$= \frac{2n}{\delta}\int_{-\frac{\pi}{n}}^{0} e^{R_m\cos\left(\theta+\frac{\pi}{n}\right)(x_0-\pi)}\,d\theta < \frac{2n}{\delta}\int_{-\frac{\pi}{n}}^{0} e^{R_m\cos\frac{\pi}{n}(x_0-\pi)}\,d\theta.$$

This exponent is negative, and so the integral tends uniformly to zero as $m\to\infty$.

This result applies to γ_m^+ immediately, since the integrand takes on along γ_m^+ values conjugate to its values on γ_m^-. Hence $\int_{\gamma_m}\frac{\delta(\varrho x)}{\varrho\delta(\varrho\pi)}\,d\varrho$ tends uniformly to zero as $m\to\infty$.

It remains to consider $\int_{\gamma_m}\frac{A}{\varrho^2\delta(\varrho\pi)}\,d\varrho$. One integral in A is

$$\int_0^x \delta_1[\varrho(x-s)]f''(s)\,ds = \sum_{k=1}^{n}\omega^{2(n-k)+1}e^{\omega^{2k-1}\rho x}\int_0^x e^{-\omega^{2k-1}\rho s}f''(s)\,ds.$$

By making a change of variable in this integral, and also in the other integral of A, certain terms will be seen to cancel one another, due to the special form of $f(x)$. The transformation is $s = s'\omega^{2(n-k+1)}$, and we have

$$\int_0^x \delta_1[\varrho(x-s)f''(s)]\,ds$$

$$= \sum_{k=1}^{n}\omega^{2(n-k)+1}e^{\omega^{2k-1}\rho x}\int_0^{\omega^{2k-2}x}\omega^{2(n-k+1)}e^{-\omega\rho s'}f''[s'\omega^{2(n-k+1)}]\,ds'.$$

But since $f(s) = \omega^{2p}f(\omega^{2p}s)$, we have $f''(s) = \omega^{6p}f''(\omega^{2p}s)$, i. e., $f''(\omega^{2p}s) = \omega^{-6p}f''(s)$. Putting this into the integral and writing s for s', we get

$$\int_0^x \delta_1[\varrho(x-s)]f''(s)\,ds = \sum_{k=1}^{n}\omega^{2k-3}e^{\omega^{2k-1}\rho x}\int_0^{\omega^{2k-2}x} e^{-\omega\rho s}f''(s)\,ds.$$

We shall take the paths of integration to be straight lines.

If we now define the point y by means of $|y| = |x|$ and $\arg y = \pi - \arg(-\omega\varrho)$ $= -\arg\omega\varrho$, we can write $\int_0^{\omega^{2k-2}x} = \int_0^{y} + \int_y^{\omega^{2k-2}x}$, this change of path of integration being allowable since the integrand is analytic throughout

a region including both paths. In fact, $f(x)$ was required to be analytic within the circle $|x| = x_0$ so that we could do this. Hence we have

$$\int_0^x \delta_1[\varrho(x-s)]\, f''(s)\, ds$$

$$= \sum_{k=1}^{n} \omega^{2k-3} e^{\omega^{2k-1}\rho x} \int_0^y e^{-\omega\rho s} f''(s)\, ds + \sum_{k=1}^{n} \omega^{2k-3} e^{\omega^{2k-1}\rho x} \int_y^{\omega^{2k-2}x} e^{-\omega\rho s} f''(s)\, ds$$

$$= \omega^{2(n-1)}\, \delta(\varrho x) \int_0^y e^{-\omega\rho s} f''(s)\, ds + \sum_{k=1}^{n} \omega^{2k-3} e^{\omega^{2k-1}\rho x} \int_y^{\omega^{2k-2}x} e^{-\omega\rho s} f''(s)\, ds.$$

The other integral in A is

$$\int_0^\pi \delta_1[\varrho(\pi-s)]\, f''(s)\, ds = \left[\int_0^x + \int_x^\pi\right] \delta_1[\varrho(\pi-s)]\, f''(s)\, ds$$

$$= \omega^{2(n-1)}\, \delta(\varrho\pi) \int_0^y e^{-\omega\rho s} f''(s)\, ds$$

$$+ \sum_{k=1}^{n} \omega^{2k-3} e^{\omega^{2k-1}\rho\pi} \int_y^{\omega^{2k-2}x} e^{-\omega\rho s} f''(s)\, ds + \int_x^\pi \delta_1[\varrho(\pi-s)]\, f''(s)\, ds.$$

Putting these back into A, the cancellation mentioned occurs, and we get

$$A = \sum_{k=1}^{n} \omega^{2k-3} \left[e^{\omega^{2k-1}\rho\pi}\, \delta(\varrho x) - e^{\omega^{2k-1}\rho x}\, \delta(\varrho\pi)\right] \int_y^{\omega^{2k-2}x} e^{-\omega\rho s} f''(s)\, ds$$

$$+ \delta(\varrho x) \int_x^\pi \delta_1[\varrho(\pi-x)]\, f''(s)\, ds,$$

or

$$\frac{A}{\varrho^2\, \delta(\varrho\pi)} = -\frac{B}{\varrho^2} + \frac{\delta(\varrho x)}{\varrho^2\, \delta(\varrho\pi)}\, C + \frac{\delta(\varrho x)}{\varrho^2\, \delta(\varrho\pi)}\, D,$$

where

$$B = \sum_{k=1}^{n} \omega^{2k-3}\, e^{\omega^{2k-1}\rho x} \int_y^{\omega^{2k-2}x} e^{-\omega\rho}\, f''(s)\, ds, \qquad C = \int_x^\pi \delta_1[\varrho(\pi-s)] f''(s)\, ds,$$

and

$$D = \sum_{k=1}^{n} \omega^{2k-3}\, e^{\omega^{2k-1}\rho\pi} \int_y^{\omega^{2k-2}x} e^{-\omega\rho s} f''(s)\, ds.$$

We show that each of the three terms of $A/\varrho^2\, \delta(\varrho\pi)$ tends uniformly to zero as $m \to \infty$. Consider first the term containing B. We have

$$\left|\omega^{2k-3}\, e^{\omega^{2k-1}\rho x} \int_y^{\omega^{2k-2}x} e^{-\omega\rho s} f''(s)\, ds\right| \leqq |e^{\omega^{2k-1}\rho x}| \int_y^{\omega^{2k-2}x} |e^{-\omega\rho s}| \cdot |f''(s)|\, |ds|.$$

As s goes from y to $\omega^{2k-2}x$, $-\omega\varrho s$ goes from $-\omega\varrho y$ to $-\omega^{2k-1}\varrho x$, i. e., from $-|\omega^{2k-1}\varrho x|$ to $-\omega^{2k-1}\varrho x$. But $R(-\omega^{2k-1}\varrho x) \geqq -|\omega^{2k-1}\varrho x|$. Hence $|e^{-\omega\rho s}|$ attains its greatest value in the range of integration at the upper limit. Hence

$$|e^{\omega^{2k-1}\rho x}|\int_y^{\omega^{2k-2}x}|e^{-\omega\rho s}|\cdot|f''(s)||ds| \leqq |e^{\omega^{2k-1}\rho x}|\cdot|e^{-\omega^{2k-1}\rho x}|\int_y^{\omega^{2k-2}x}|f''(s)||ds|.$$

Hence, if $|f''(s)| \leqq M$, a constant, in the circle $|x| = x_0$, we have $|B| < 2Mnx_0$. Hence $\int_{\gamma_m}\frac{B}{\varrho^2}d\varrho$ converges uniformly to zero as $m \to \infty$.

The term containing C is $\frac{\delta(\varrho x)}{\varrho^2\delta(\varrho\pi)}\int_x^{\pi}\delta_1[\varrho(\pi-s)]f''(s)\,ds$. This expression takes on along γ_m^+ values conjugate to its values along γ_m^-; we need therefore consider it only along γ_m^-. We have

$$|\delta_1[\varrho(\pi-s)]| \leqq \sum_{k=1}^{n}|e^{\omega^{2k-1}\rho(\pi-s)}| = \sum_{k=1}^{n}e^{\left[\xi\cos\frac{(2k-1)\pi}{n}-\eta\sin\frac{(2k-1)\pi}{n}\right](\pi-s)}.$$

But we had

$$\xi\cos\frac{\pi}{n}-\eta\sin\frac{\pi}{n} \geqq \xi\cos\frac{(2k-1)\pi}{n}-\eta\sin\frac{(2k-1)\pi}{n}$$

on γ_m^- for $k = 1, 2, \cdots n$. Hence

$$|\delta_1[\varrho(\pi-s)]| \leqq ne^{\left(\xi\cos\frac{\pi}{n}-\eta\sin\frac{\pi}{n}\right)(\pi-s)} \leqq ne^{\left(\xi\cos\frac{\pi}{n}-\eta\sin\frac{\pi}{n}\right)(\pi-x)},$$

since $x \leqq s$ in the term containing C. Hence

$$|C| \leqq \int_x^{\pi}|\delta_1[\varrho(\pi-s)]|\cdot|f''(s)|\,ds \leqq n\pi M|e^{\omega\rho(\pi-x)}|.$$

But we have also on γ_m^- for sufficiently large values of m $\left|\frac{\delta(\varrho x)}{\delta(\varrho\pi)}\right| \leqq \frac{2n}{\delta}|e^{\omega\rho(x_0-\pi)}|$, where $0 \leqq x \leqq x_0 < \pi$. Hence

$$\left|\frac{\delta(\varrho x)}{\delta(\varrho\pi)}C\right| \leqq \frac{2\pi Mn^2}{\delta}|e^{\omega\rho(x_0-x)}| \leqq \frac{2\pi Mn^2}{\delta}.$$

Hence $\int_{\gamma_m^-} \frac{\delta(\varrho x)}{\varrho^2 \delta(\varrho \pi)} C d\varrho$ tends uniformly to zero as $m \to \infty$. As we saw above, the same is true of the integral along γ_m^+. Hence $\int_{\gamma_m} \frac{\delta(\varrho x)}{\varrho^2 \delta(\varrho \pi)} C d\varrho$ tends uniformly to zero as $m \to \infty$.

We have finally to consider the integral containing D. We have

$$|D| \leqq \sum_{k=1}^{n} \left| e^{\omega^{2k-1}\rho\pi} \int_y^{\omega^{2k-2}x} e^{-\omega\rho s} f''(s)\, ds \right|.$$

Using the results obtained in the consideration of B, we get

$$|D| \leqq 2Mx_0 \sum_{k=1}^{n} \left| e^{\omega^{2k-1}\rho(\pi-x)} \right| \leqq 2Mnx_0 \left| e^{\omega\rho(\pi-x)} \right|$$

if we restrict ourselves to γ_m^-. Hence, as in the case of C, $\int_{\gamma_m^-} \frac{\delta(\varrho x)}{\varrho^2 \delta(\varrho \pi)} D d\varrho$ tends uniformly to zero as $m \to \infty$. The integral over γ_m^+ is treated similarly. If ϱ is on γ_m^+, we have $\xi \sin \frac{(k-1)\pi}{n} \geqq -\eta \cos \frac{(k-1)\pi}{n}$, by arguments like those used on pages 31 and 32. But from this we deduce

$$\xi \cos \frac{(2n-1)\pi}{n} - \eta \sin \frac{(2n-1)\pi}{n} \geqq \xi \cos \frac{(2k-1)\pi}{n} - \eta \sin \frac{(2k-1)\pi}{n}.$$

Hence $R(\omega^{2n-1}\varrho) \geqq R(\omega^{2k-1}\varrho)$ for $k = 1, 2, \cdots n$, so that $\left|e^{\omega^{2n-1}\rho(\pi-x)}\right| \geqq \left|e^{\omega^{2k-1}\rho(\pi-x)}\right|$ for these values of k. Hence on γ_m^+ we have $|D| \leqq 2Mnx_0 \left|e^{\omega^{2n-1}\rho(x-\pi)}\right|$. But $\left|\frac{\delta(\varrho x)}{\delta(\varrho \pi)}\right| \leqq \frac{2n}{\delta} \left|e^{\omega^{2n-1}\rho(x-\pi)}\right|$ provided ϱ is on γ_m^+. Hence $\left|\frac{\delta(\varrho x)}{\delta(\varrho \pi)} D\right| \leqq \frac{4Mn^2x_0}{\delta}$, so that $\int_{\gamma_m^+} \frac{\delta(\varrho x)}{\varrho^2 \delta(\varrho \pi)} D d\varrho$ converges uniformly to zero as $m \to \infty$. This completes the proof of the theorem.

MAXIMAL CUSPIDAL CURVES.

TEMPLE RICE HOLLCROFT.

1. **Introduction.** It has been long known that for $n>4$ all the double points of a plane algebraic curve can not be cusps, but definite results have not been obtained until recently. In 1864, Clebsch* determined the maximum number of cusps for a rational curve. Forty years later, Wieleitner† found an upper bound to the number of cusps for a given order n to be $3n(n-2)/8$ from the Plücker formula for the number of inflections and the fact that this number can not be negative.

In 1913, Lefschetz‡ found the exact upper limit imposed by Plücker's equations to be $n(n-2)/3$ from the fact that the number of bitangents can not be negative. In the first part of his paper, he proves that the number of independent absolute invariants relations among the coefficients of the equation of a curve of order n and genus $p \geqq 2$ can not exceed $\frac{1}{2}n(n+3)-8$ and that each node accounts for one and only one such invariant. He then states his Postulate of Singularities upon which limits (II) and (3), given below, rest. This postulate assumes that the change of one node into a cusp always causes the addition of one and only one independent absolute invariant. The postulate may be stated as follows:

$$F = \tfrac{1}{2}n(n+3)-\delta-2\varkappa-8 = 3n+p-9-\varkappa, \quad p \geqq 2,$$

wherein F is the number of independent absolute invariants among the coefficients of the equation of a curve of order n and genus p with $\varkappa$ cusps and δ nodes. Using this postulate, Lefschetz finds that for $n \geqq 14$, the fact that $F \geqq 0$ gives a more restrictive limit on the cusps than Plücker's equations. He also derives formulas for the maximum numbers of cusps that a curve of given order n and genus p may have, the first two from Plücker's equations only and the third involving the additional condition $F \geqq 0$. His results are as follows:

The maximum number of cusps $\varkappa$ for a given n:

(I) For $n \leqq 13$, $[\varkappa]$§ $\leqq \frac{1}{8}n(n-2)$

(II) For $n \geqq 14$, $[\varkappa] \leqq \frac{1}{4}n(n+3)-4$.

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* A. Clebsch: Crelle's Journal, vol. 64 (1864), p. 51.

† H. Wieleitner: Theorie der ebenen algebraischen Kurven höherer Ordnung, (1905), p. 76.

‡ S. Lefschetz: "On the existence of loci with given singularities," Trans. Amer. Math. Soc., vol. 14 (1913), pp. 23–41.

§ The symbol $[x]$ followed by $\leqq$ ($\geqq$) means the largest (one greater than the largest if not equal) integer contained in the expression on the right.

The maximum number of cusps for a given n and p: (Lefschetz has stated the limits within which these formulas hold in terms of n expressed as a function of a given p. For the purpose of this paper, it is more desirable to consider n as given and to state the limits in terms of p expressed as a function of the given n).

For $0 \leqq p \leqq \frac{1}{2}(n+2-\sqrt{4n+13})$,

$$[\varkappa] \leqq \tfrac{3}{2}(n+2p-2); \tag{1}$$

For $\frac{1}{2}(n+2-\sqrt{4n+13}) < p \leqq \frac{1}{2}(2n-1+\sqrt{16n-23})$,

$$[\varkappa] \leqq 2(n+p)-\tfrac{1}{2}(11+\sqrt{24p-8n+25}); \tag{2}$$

For $\frac{1}{2}(2n-1+\sqrt{16n-23}) \leqq p \leqq \frac{1}{4}(n-4)(n-5)$,

$$\varkappa = 3n+p-9; \tag{3}$$

For $\frac{1}{4}(n-4)(n-5) \leqq p \leqq \frac{1}{2}(n-1)(n-2)$,

$$\varkappa = \tfrac{1}{2}(n-1)(n-2)-p. \tag{4}$$

Formula (4) is merely the statement that for $p \geqq (n-4)(n-5)/4$, the maximum number of cusps is limited only by p.

Lefschetz, in this same paper, and in 1922, Coolidge* proved that curves with any number of cusps up to and including the maximum, as given by formula (1) above, exist. Formal proof has not been given for the others, but their existence is practically assured.

In 1917, Haskell,† using the same method as Lefschetz, found the upper limit $n(n-2)/3$ to the number of cusps for a given n from Plücker's equations. He also proved that the curves with the maximum number of cusps for a given n, as determined in this way, are self dual. His results are as follows:

For n of the form $3a$ or $3a+2$ (a any positive integer)

$$m = n, \qquad p = \tfrac{1}{6}(n-2)(n-3),$$

$$\delta = \tau = 0, \qquad \varkappa = \iota = \tfrac{1}{2}n(n-2);$$

* J. L. Coolidge: "On the existence of curves with assigned singularities", Bull. Am. Math. Soc., vol. 28 (1922), pp. 451–455.

† M. W. Haskell: "The maximum number of cusps of an algebraic plane curve and enumeration of self dual curves", Bull. Am. Math. Soc., vol. 23 (1917), pp. 164–165.

For n of the form $3a+1$,

$$m = n, \qquad p = \tfrac{1}{6}(n-1)(n-4),$$
$$\delta = \tau = 1, \qquad \varkappa = \iota = \tfrac{1}{3}(n^2-2n-2).$$

(The symbols m, τ, ι, denote the class, number of bitangents, number of inflections, respectively.)

According to the Lefschetz postulate, for $n \geqq 14$, the foregoing self dual curves do not exist.

In 1923, the author* solved the following problems:

I. To find the greatest number of cusps that may be added to the singularities of a plane curve of order n with $\varkappa_1$ cusps and δ_1 nodes.

II. To find the greatest number of nodes that may be added to the singularities of a plane curve of order n with $\varkappa_1$ cusps and δ_1 nodes.

III. To find the greatest number of cusps that may occur among the remaining singularities of a plane curve of order n and genus p with $\varkappa_1$ cusps and δ_1 nodes.

IV. To find the least order n of a plane curve that can possess a given number $\varkappa_1$ of cusps and δ_1 of nodes.

The methods employed in solving these problems are so similar to those used in the present paper, that in most cases only an outline of the method will be given.

2. **Definitions and purpose.** Two curves of the same order and with the same singularities are said to be of the same type. Since in this paper, curves are to be discussed only with respect to the properties determining its type, any number of curves of the same type will be spoken of as one and the same curve.

When the maximum number of cusps for a given order and genus is found by the preceding formulas, the remaining Plücker characteristics follow. Then for a given order and genus, there is one and only one curve with the maximum number of cusps for that order and genus. Such a curve will be called a *maximal cuspidal curve.*

When the order is given and the cusps are limited by other than the genus, for every class there exists at least one maximal cuspidal curve, for some classes only one and for others, more than one.

The purpose of the present paper is to discuss the curves having $\varkappa$ as a maximum for a given n and p as they occur within all classes for the given n and to ascertain the number of such curves for each class.

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*T. R. Hollcroft, "Singularities of curves of given order", Bull. Am. Math. Soc., vol. 29 (1923), pp. 407–414.

3. Curves of minimum class m_0. The least class m_0 for a given n occurs when both τ and ι are as near zero as possible. Assuming both zero, we have from Plücker's equation,

$$m_0(m_0-1) \geqq n$$

or, on solving for m_0,

$$[m_0] \geqq \tfrac{1}{2}(1+\sqrt{4n+1}).$$

For n even, $p=(n-2m+2)/2$ when $\iota = 0$. In this substitute for m the second member of the inequality defining m_0 and we obtain

$$[p_0] \leqq \tfrac{1}{2}(n+1-\sqrt{4n+1}).$$

Similarly for n odd,

$$[p_0] \leqq \tfrac{1}{2}(n+2-\sqrt{4n+13}).$$

For $n>6$, the second inequality should be used to define the least genus p_0 of curves of class m_0.

Since, for a given n, the minimum class m_0 is found by making both ι and τ as near zero as possible, the maximum value of $\varkappa$ occurs for that class. Then all curves of minimum class m_0 are maximal cuspidal curves.

For all curves of class m_0 and genus p_0, ι is 0 or 1 for n even or odd respectively, but τ may have values ranging from 0 to m_0-2. For $m=m_0$, if p is allowed to increase by unity, $\varkappa$ and ι each increase 2 and δ and τ each decrease 3. Since for all curves of class m_0, $\delta>\tau$, the number of maximal cuspidal curves of class m_0 is governed by the value of τ_0, the number of bitangents of the curve of least genus p_0 of class m_0. Then in order to find the number of curves of class m_0 for a given n, we must ascertain the value of τ_0 for that n. To do this, we have for n even [odd],

$$n = m_0(m_0-1)-2\tau_0[-3].$$

For any n (except $n=5$ for which $m_0=4$) determine m_0 from

$$[m_0] \geqq \tfrac{1}{2}(1+\sqrt{4n+1}),$$

and the value of τ_0 is given for n even or odd respectively by

$$\tau_0 = \tfrac{1}{2}[m_0(m_0-1)-n]$$

or

$$\tau_0 = \tfrac{1}{2}[m_0(m_0-1)-n-3].$$

For any n the number α defined by

$$[\alpha] \geqq \tfrac{1}{3}(r_0+1)$$

is the number of curves of minimum class m_0 ranging in genus from p_0 to $p_0+\alpha-1$. As shown above, these α curves are all maximal cuspidal curves. The number α depends somewhat on the size of n, but even more so on the proximity of n to the product of two successive integers. For any value of n that is the product of two successive integers, $\alpha = 1$.

4. **Curves of least class m_1 for which $F = 0$.** For $n \geqq 14$, as proved by Lefschetz, the maximum number of cusps for a given n is determined by the number of independent absolute invariants. For a given $n \geqq 14$, we desire now to find the least class m_1 for which this limit supersedes that given by Plücker's equations. To do this, we first find p_1, the genus of class m_1 for which this occurs, by eliminating $\varkappa$ from the two expressions:

$$[\varkappa] \leqq 2(n+p)-\tfrac{1}{3}(11+\sqrt{24p-8n+25})$$

$$\varkappa = 3n+p-9,$$

which gives

$$[p_1] \geqq \tfrac{1}{3}(2n-1+\sqrt{16n-23}).$$

Substitute either of the above limits for $\varkappa$ and this value of p_1 in the formula

$$m = 2(n+p-1)-\varkappa$$

and we obtain

$$[m_1] \geqq \tfrac{1}{3}(13+\sqrt{16n-23}).$$

There may be more than one maximal cuspidal curve of class m_1, but there is only one of genus p_1 and that is the one that lies on the boundary between the maximal cuspidal curves for $p \leqq p_1$ for which the maximum number of cusps is determined by Plücker's equations only, and for $p \geqq p_1$ for which the maximum number of cusps is determined by the condition $F \geqq 0$. If there are more than one maximal cuspidal curve of class m_1, they will be of genus $p < p_1$ and will belong to the group for which the maximum number of cusps is determined by Plücker's equations only.

5. **Curves of class m such that $m_0 < m \leqq m_1$.** In finding the number of maximal cuspidal curves of each class m such that $m_0 < m \leqq m_1$, we begin with the self dual maximal cuspidal curves whose characteristics, as found by Haskell, are given in section 1, and allow p to decrease, as-

certaining in each case how many times p may decrease by unity before m decreases by unity. In order to do this, we must know how the singularities vary as p decreases. For $p > p_0$, $\varkappa$ decreases 0, 1 or 2 for each decrease of p by unity and the other singularities vary as follows:

If								
$\varkappa$	decreases	0	$\varkappa$	decreases	1	$\varkappa$	decreases	2
δ	increases	1	δ	increases	2	δ	increases	3
ι	decreases	6	ι	decreases	4	ι	decreases	2
m	decreases	2	m	decreases	1	m	decreases	0
τ	decreases	$2(m-6)$	τ	decreases	$m-7$	τ	increases	3.

As to the first column, it can be proved that when p decreases by unity, $\varkappa$ remains constant only for $n \leqq 20$ and $m = 6$ or 7. In all, this can occur for only nine curves, viz., for $m = 6$, $n = 7, 9, 10, 12$, and for $m = 7$, $n = 8, 11, 14, 17, 20$. Only the changes given in the last two columns need therefore be considered in the following general deductions.

Starting with the maximal cuspidal curve for n of form $3a$ and of highest genus $p = (n-2)(n-3)/6$, so long as $\varkappa$ decreases 2 for each decrease of one in p, the curve remains self dual. In order to find the first curve for which $m = n-1$, we must find the first curve of genus $p'-1$ for which the number of cusps is one less than for the curve of genus p'. This curve will be of class $n-1$. Having found the characteristics of this curve, repeat the above process and find the characteristics of the maximal cuspidal curve of highest genus for the class $n-2$, etc. The limiting values of p are found from the fact that τ can not have negative values. The details are not given, since the methods imployed are similar in plan to those used in a previous paper.*

The process outlined above is repeated until the law governing the formation of the sequence giving the number of maximal cuspidal curves in each class for each of the three forms of n is discovered. The resulting formulas for the number of maximal cuspidal curves in each class m, such that $n \geqq m > m_0$, are given in the following table:

Class		n	$n-1$	$n-2$	$n-3$	$n-4$	$n-5$	$n-6$	$n-7$	...
Number of maximal cuspidal curves of given class for n of form	$3a$	$\frac{1}{3}(n-3)$	$\frac{1}{3}(n-6)$	$\frac{1}{3}(n-6)$	$\frac{1}{3}(n-6)$	$\frac{1}{3}(n-9)$	$\frac{1}{3}(n-9)$	$\frac{1}{3}(n-9)$	$\frac{1}{3}(n-12)$	...
	$3a+1$	$\frac{1}{3}(n-4)$	$\frac{1}{3}(n-4)$	$\frac{1}{3}(n-7)$	$\frac{1}{3}(n-7)$	$\frac{1}{3}(n-7)$	$\frac{1}{3}(n-10)$	$\frac{1}{3}(n-10)$	$\frac{1}{3}(n-10)$	...
	$3a+2$	$\frac{1}{3}(n-2)$	$\frac{1}{3}(n-5)$	$\frac{1}{3}(n-8)$	$\frac{1}{3}(n-5)$	$\frac{1}{3}(n-8)$	$\frac{1}{3}(n-11)$	$\frac{1}{3}(n-8)$	$\frac{1}{3}(n-11)$	...

* T. R. Hollcroft, loc. cit.

In general, for n of form $3a$ there are $(n-3q-3)/3$ maximal cuspidal curves for each of the classes $n-3q+2$, $n-3q+1$, $n-3q$; and for n of form $3a+1$, $(n-3q-4)/3$ for each of the classes $n-3q+1$, $n-3q$, $n-3q-1$. For n of form $3a+2$ there are respectively for the classes

$$\begin{array}{lll} n-3q+2, & n-3q+1, & n-3q, \\ \frac{1}{3}(n-3q-2), & \frac{1}{3}(n-3q-5), & \frac{1}{3}(n-3q-2), \end{array}$$

maximal cuspidal curves.

The above formulas give the number of maximal cuspidal curves for each class down to but not including the minimum class m_0. This is to be expected from the fact that these formulas depend only on the numerical value of n while the number of curves for the class m_0 may be one or more than one, whatever the value of n. The class m_0 is at the point of division between classes which have only one maximal cuspidal curve for each class, and those which may have many for each class.

Since all of the curves to which the above formulas apply are of genus $p > p_0$, they are not included in the existence proofs given by Lefschetz* and Coolidge.*

In the following, the term "exist" is used to mean that the characteristics of the curves satisfy Plücker's equations and the Lefschetz postulate.*

Curves exist whose numbers for the respective classes are given by these formulas for all values of $n \leqq 13$. For $n \geqq 14$, since $F \geqq 0$, such maximal cuspidal curves exist only for the classes m such that $m \leqq m_1$.

Since for $n \leqq 13$, $m_1 > n$, it can be stated in general that maximal cuspidal curves exist whose numbers are given by these formulas for all values of n and for m such that $m_0 < m \leqq m_1$.

The class m_1 is the largest class for which the *full number* of maximal cuspidal curves given in the table of this section exist. In addition, some number less than the full number exist for class m_1+1. To determine these, first find p_1 for a given n. If the maximal cuspidal curve of genus p_1 is of class m_1+1, then that curve and all such curves of lower genus of class m_1+1 exist.

When the genus p_1 does not belong to a curve of class m_1+1, it must belong to the curve of highest genus of class m_1. Since when $p \geqq p_1$, $\varkappa = 3n+p-9$, for $p \geqq p_1$, the number of cusps increases one for each increase of p by unity. We have found in this section that for maximal cuspidal curves satisfying Plücker's equations, $\varkappa$ increases one for an increase of p by unity when also m increases by unity. Then, if for a given n, p_1 is the genus of the curve of highest genus in class m_1, and then only,

* Cf. references, Section 1.

the curve of lowest genus p_1+1, and that one only, of class m_1+1 also exists. As an example of this, for $n=15$ we find $p_1=22$, which is the highest genus for class $m_1=14$. Then the maximal cuspidal curve for which $n=m=15$, $p=23$ exists.

6. **Self dual maximal cuspidal curves.** The curves whose maximum number of cusps for a given n is determined by Plücker's equations only, are self dual and exist only for $n \leqq 13$. In addition to these, for $n \geqq 8$ other self dual maximal cuspidal curves appear for p within the limits $\frac{1}{6}(n^2-7n+16)$ and $\frac{1}{6}(n-2)(n-3)$ inclusive. In order to find the largest value of n for which these curves exist, solve simultaneously the relations,

$$[p] \geqq \tfrac{1}{6}(n^2-7n+16), \quad \varkappa = 2n+p-2, \quad \varkappa = 3n+p-9,$$

which gives $n=m=15$, $p=23$, $\varkappa=\iota=59$, $\delta=\tau=9$.

As shown above, this is the only self dual maximal cuspidal curve of order 15 that exists. For $p=22$, no self dual maximal cuspidal curve exists, but for all other values of p such that $0 \leqq p \leqq 23$, self dual maximal cuspidal curves exist, one and only one for each value of p. Thus there exist in all, twenty-three such curves, the upper limit of whose cusps is determined by Plücker's equations only, one curve for each value of n such that $3 \leqq n \leqq 7$ except $n-4$; two each for $8 \leqq n \leqq 10$; three each for $11 \leqq n \leqq 14$ and one for $n=15$.

The simplest self dual maximal cuspidal curve, the cubic with one cusp is easily constructed. Next to the cuspidal cubic, the simplest is the del Pezzo quintic, the existence of which has been established by del Pezzo* and curves for which the positions of the five cusps are given have been constructed by Field.† These two are the only curves of the twenty-three whose existence has been rigorously established. The fact that these can be constructed implies that the others can also, but for $n>5$ the algebraic difficulties involved in an actual construction are very great, and in addition, the curves become increasingly indeterminate.

In addition to the above twenty-three self dual maximal cuspidal curves, the upper limit to whose cusps is determined by Plücker's equations, for $n \geqq 16$ self dual maximal cuspidal curves exist, one and only one for each value of n. The upper limit to the number of cusps of these curves is determined by the Lefschetz postulate. As will be shown in the next section, for $p \geqq p_1$ there exists one and only one maximal cuspidal curve for each

* Pasquale del Pezzo, Rendiconti dell' Accad. delle Scienze fisiche e matematiche di Napoli, Series 2, Vol. III (1889), pp. 46–49.

† Peter Field: "On the form of a plane quintic curve with five cusps", Trans. Amer. Math. Soc., Vol. 7 (1906), pp. 26–32.

class. When $m_1 < n$, which is true for all values of $n \geqq 16$, one of these curves for each value of n will be of class n.

In general, the value of $\varkappa$ may be found for such self dual curves by eliminating δ from the equations,

$$2\delta + 3\varkappa = n(n-1) - n,$$

$$\delta + 2\varkappa = \tfrac{1}{2}n(n+3) - 8,$$

which gives

$$\varkappa = 5n - 16.$$

Substituting this value of $\varkappa$ in the formula

$$\varkappa = 3n + p - 9,$$

we find

$$p = 2n - 7.$$

Then for $n \geqq 16$, self dual maximal cuspidal curves have

$$\varkappa = 5n - 16, \qquad p = 2n - 7,$$

and one and only one such curve exists for each value of n. The self dual maximal cuspidal curves for $n = 14$, $p = 21$ and $n = 15$, $p = 23$ are common to both this set and the one preceding. The formula for p also shows that for $p > 20$, self dual maximal cuspidal curves occur for every odd value of p and for no even values.

7. **Curves of class $m < m_0$, $p < p_0$, and $m > m_1$.** For curves of class $m < m_0$ when $p < p_0$, the maximum number of cusps occurs when ι is a minimum. This minimum is 0 for n even or 1 for n odd. For these the maximum number of cusps for a given n and p is

$$[\varkappa] \leqq \tfrac{3}{2}(n + 2p - 2).$$

Beginning with $p = 0$, for each increase of p by unity up to and including p_0, $\varkappa$ increases 3, δ increases 4, ι does not vary, τ decreases m—1, m decreases 1. Beginning with the value $m = (n+2)/2$ for $p = 0$, m decreases 1 for each increase of p by unity down to m_0 and thus there exists one and only one maximal cuspidal curve for each class such that $\frac{1}{2}(n+2) \leqq m < m_0$. These same values of m are repeated for $p > p_0$, where, in general, more than one maximal cuspidal curve exists for each class as shown in section 5.

For $m > m_1$, the maximum number of cusps is determined by the Lefschetz postulate to be

$$\varkappa = 3n + p - 9.$$

The class of a maximal cuspidal curve whose cusps are given by this formula is

$$m = p - n + 7.$$

The maximum number of cusps for a given $n \geqq 14$ is

$$[\varkappa] \leqq \tfrac{1}{4}n(n+3) - 4,$$

and for such a curve

$$[p] \leqq \tfrac{1}{4}(n-4)(n-5), \qquad [m] \leqq \tfrac{1}{4}n(n-13) + 12.$$

When the equality sign holds for $\varkappa$ it holds also for p and m and $\delta = 0$. When the inequality sign holds for $\varkappa$ it holds also for p and m and $\delta = 1$.

For each increase of p by unity from p_1 up to $(n-4)(n-5)/4$, m increases by unity from m_1 (or $m_1 + 1$) up to $n(n-13)/4 + 12$ and there is thus one and only one maximal cuspidal curve for each value of $m \geqq m_1$.

Then for $p \geqq p_1$ as well as for $p < p_0$, there is one and but one maximal cuspidal curve for each class, but with this difference. For $p < p_0$, a maximal cuspidal curve has no definite properties aside from its Plücker numbers, but for $p \geqq p_1$, the curve as a whole is invariant under a collineation, that is, all of its projective properties are fully determined.

For $p \geqq (n-4)(n-5)/4$, the genus is the only limit on the cusps, and the maximal cuspidal curves, one for each set of values of n and p, have no nodes. For each increase of p by unity, $\varkappa$ decreases 1 and m increases 3 so that there is a maximal cuspidal curve only for every third value of the class for a given n.

WELLS COLLEGE.

PLANE CUBICS ASSOCIATED WITH THE QUADRANGLE-QUADRILATERAL CONFIGURATION.

By B. M. Turner.

1. **Introduction.** It is known that in the special Desargues configuration 10_3, called the quadrangle-quadrilateral configuration,* the associated four-point and four-line are each covariant to the other. In a former paper† by the writer it was shown that four real points chosen arbitrarily in the plane as critic centers (vertices of inflexional triangles), determine a syzygetic pencil of cubics as one of four. It appears that this quadruple of pencils of cubics is projectively associated with the quadrangle-quadrilateral configuration. The points and lines of the 10_3 can be interpreted in terms of the cubics, the points of inflexion and critic centers for the cubics have unique positions on the ten lines, and the four Hessian configurations $(9_4\ 12_3)$ unite into a configuration $(18_7\ 42_3)$.

It further appears that the points and lines of the quadrangle-quadrilateral configuration, together with those of the diagonal triangle common to the associated 4-point and 4-line, can be interpreted in terms of a second quadruple of pencils of cubics also projectively associated with the 10_3.

Furthermore, the points of inflexion of the four sets of thirteen pencils of cubics, i. e., of the four configurations C_{13}‡, of which each of the quadruples is the origin, all lie on the configuration and diagonal lines of the 10_3.

Klein's group G_{41} which carries the 4-point and 4-line into themselves, also leaves invariant each of the quadruples of pencils of cubics, and each of the sets of four configurations C_{13}.

2. **The quadrangle-quadrilateral configuration in terms of cubics.** Four real points P, $(1, \pm 1, \pm 1)$, chosen arbitrarily in the plane, determine six lines l,

$$y \pm z = 0, \qquad z \pm x = 0, \qquad x \pm y = 0.$$

In accordance with the results of the paper§, any three of the points taken as vertices of the real inflexional triangle and the fourth as the

* Kantor: Die Configurationen $(3,3)_{10}$, (B); Wiener Sitzber., Bd. 84 (1881), p. 1291.
Morley: Projective Coördinates; Trans. Amer. Math. Soc., vol. 4 (1903), p. 291.
H. F. MacNeish: Dissertation, University of Chicago (1912), p. 14.
Veblen & Young: Projective Geometry, vol. 1, p. 46.

† Annals of Math., vol. 23 (1922) pp. 287-291.

‡ The configuration C_{13} is fully discussed in a paper not yet published, but enough detail is given in this paper to make it comprehensible to the reader.

§ Annals of Math., loc. cit.

point common to the three real harmonic polars, uniquely determine a pencil of cubics; and the three real harmonic polars are the lines joining the fourth point to each of the other three. This may be stated as

THEOREM I. *The six sides of a real 4-point in four ways form a triangle and three concurrent lines, such that each set of three concurrent lines are the real harmonic polars for a pencil of cubics having the other three of the six lines as sides of its real inflexional triangle.*

The 4-point uniquely determines a 4-line

$$x \pm y \pm z = 0,$$

each line the polar of one point P with respect to the triangle formed by the other three, such that the associated 4-point and 4-line form the special Desargues configuration 10_3, known as the quadrangle-quadrilateral configuration. The six vertices I,

$$(0,\ 1,\ \pm 1), \qquad (\pm 1,\ 0,\ 1), \qquad (1,\ \pm 1,\ 0),$$

of the 4-line, because of the polarity properties, as they lie by threes on the lines, are the real points of inflexion for the pencils of cubics.

THEOREM II. *The four vertices of a real 4-point are critic centers for four pencils of cubics which have the six vertices of the associated 4-line, in the quadrangle-quadrilateral configuration, as the real points of inflexion; and the ten lines of the configuration as the real inflexional axes.*

3. Eight imaginary critic centers and imaginary inflexional axes. The fundamental conic K,

$$x^2 + y^2 + z^2 = 0,$$

of the polarity established by the 4-point and 4-line, intersects the four lines in the points Ω,

$$(1,\ \pm\omega,\ \pm\omega^2), \qquad (1,\ \pm\omega^2,\ \pm\omega), \qquad [\omega^3 = 1].$$

The tangents to K at the points Ω are the lines Ψ,

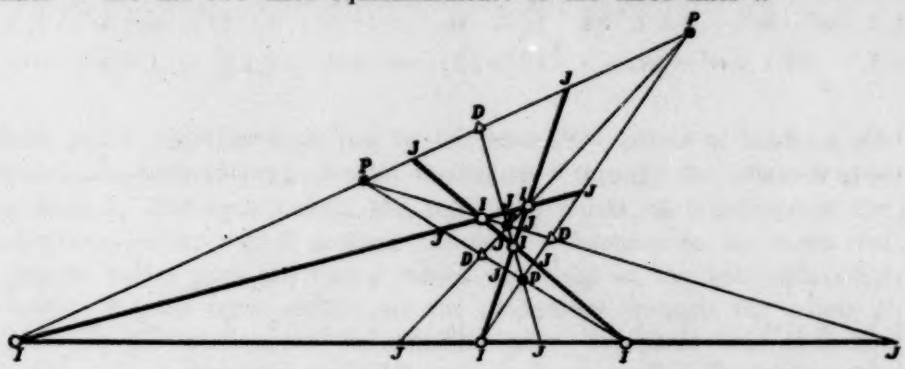
$$x \pm \omega y \pm \omega^2 z = 0, \qquad x \pm \omega^2 y \pm \omega z = 0.$$

The points Ω and lines Ψ are critic centers and inflexional axes for the quadruple of pencils of cubics,

$$(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)+\lambda(y+z)(z+x)(x+y)=0,$$
$$(-x+y+z)(-x+\omega y+\omega^2 z)(-x+\omega^2 y+\omega z)+\lambda(y+z)(z-x)(-x+y)=0,$$
$$(x-y+z)(x-\omega y+\omega^2 z)(x-\omega^2 y+\omega z)+\lambda(-y+z)(z+x)(x-y)=0,$$
$$(x+y-z)(x+\omega y-\omega^2 z)(x+\omega^2 y-\omega z)+\lambda(y-z)(-z+x)(x+y)=0.$$

THEOREM III. *The fundamental conic of the polarity established by the associated* 4*-point and* 4*-line, in the quadrangle-quadrilateral configuration, intersects the sides of the* 4*-line in eight imaginary critic centers for the four pencils of cubics which have the vertices of the* 4*-point as real critic centers; and the tangents to the conic at the eight points are eight imaginary inflexional axes for the cubics.*

On each side of the 4-line, the points Ω are the two points equianharmonic to the three points I; and through each of the points P, the lines Ψ are the two lines equianharmonic to the three lines l.



THEOREM IV. *The four pencils of cubics with the vertices of a real* 4*-point as critic centers, have, as imaginary critic centers, the points equianharmonic to each of the four sets of three collinear vertices of the associated* 4*-line in the quadrangle-quadrilateral configuration; and, as imaginary inflexional axes, the lines equianharmonic to each of the four sets of three concurrent sides of the* 4*-point.*

4. **The inflexional set of points.** The diagonal triangle common to the 4-point and 4-line has vertices D,

$$(1,\ 0,\ 0),\qquad (0,\ 1,\ 0),\qquad (0,\ 0,\ 1).$$

The imaginary points of inflexion for the four pencils of cubics are the intersections of the lines l by the lines Ψ,

$$(\pm 1,\ \pm 1,\ \omega-\omega^2),\quad (\pm 1,\ \omega-\omega^2,\ \pm 1),\quad (\omega-\omega^2,\ \pm 1,\ \pm 1).$$

One pair of these points lies on each side of the 4-point equianharmonic to the points P and D.

THEOREM V. *The four pencils of cubics determined by the vertices of a real 4-point as critic centers, have, as imaginary points of inflexion, the two points on each of the six sides of the 4-point equianharmonic to the vertices and diagonal point.*

The points of inflexion for the four pencils of cubics, arranged in the well known scheme to display the collinearity, are

$$\begin{array}{lll}
(0, 1, -1), & (\omega^2-\omega, 1, -1), & (\omega-\omega^2, 1, -1),\\
(-1, \omega-\omega^2, 1), & (-1, 0, 1), & (-1, \omega^2-\omega, 1),\\
(1, -1, \omega^2-\omega), & (1, -1, \omega-\omega^2), & (1, -1, 0);
\end{array}$$

$$\begin{array}{lll}
(0, 1, -1), & (\omega^2-\omega, 1, -1), & (\omega-\omega^2, 1, -1),\\
(1, \omega^2-\omega, 1), & (1, 0, 1), & (1, \omega-\omega^2, 1),\\
(1, 1, \omega^2-\omega), & (1, 1, \omega-\omega^2), & (1, 1, 0);
\end{array}$$

$$\begin{array}{lll}
(0, 1, 1), & (\omega^2-\omega, 1, 1), & (\omega-\omega^2, 1, 1),\\
(-1, \omega-\omega^2, 1), & (-1, 0, 1), & (-1, \omega^2-\omega, 1),\\
(1, 1, \omega-\omega^2), & (1, 1, \omega^2-\omega), & (1, 1, 0);
\end{array}$$

$$\begin{array}{lll}
(0, 1, 1), & (\omega-\omega^2, 1, 1), & (\omega^2-\omega, 1, 1),\\
(1, \omega-\omega^2, 1), & (1, 0, 1), & (1, \omega^2-\omega, 1),\\
(1, -1, \omega^2-\omega), & (1, -1, \omega-\omega^2), & (1, -1, 0).
\end{array}$$

These sets of points show relations which may be expressed as

THEOREM VI. *Four real points chosen arbitrarily in the plane as critic centers determine a quadruple of pencils of cubics with an inflexional set of six real and twelve imaginary points: each two of the four pencils have in common one real and two imaginary points of inflexion; and each of the eighteen points belongs to the inflexional sets for two, and only two, of the four pencils.*

5. **The configuration** $(18_7\ 42_3)$. In addition to the ten real and eight imaginary lines named in the preceding, the complete set of inflexional axes includes

$$\begin{array}{ll}
\pm x \pm y + 2\omega z = 0, & \pm x \pm y + 2\omega^2 z = 0,\\
\pm x + 2\omega y \pm z = 0, & \pm x + 2\omega^2 y \pm z = 0,\\
2\omega x \pm y \pm z = 0, & 2\omega^2 x \pm y \pm z = 0;
\end{array}$$

that is, each two of the four pencils have one real inflexional axis in common; and in all there are forty-two inflexional axes, ten real and thirty-two imaginary.

The schematic arrangement of the points of the four inflexional sets shows that the eighteen points of inflexion lie by threes on the forty-two inflexional axes, and that the forty-two axes pass by sevens through the eighteen points.

THEOREM VII. *The combination of the four Hessian configurations* $(9_4\ 12_3)$ *for the four pencils of cubics determined by four real points as critic centers is a configuration* $(18_7\ 42_3)$.

6. **The complete set of critic centers and points of inflexion.** The remaining critic centers are

$$\begin{array}{llll} (\pm 1, & \pm 1, & 2\omega - 1), & (\pm 1, \quad \pm 1, \quad 2\omega^2 - 1), \\ (\pm 1, & 2\omega - 1, & \pm 1), & (\pm 1, \quad 2\omega^2 - 1, \quad \pm 1), \\ (2\omega - 1, & \pm 1, & \pm 1), & (2\omega^2 - 1, \quad \pm 1, \quad \pm 1). \end{array}$$

These points together with the twelve imaginary points of inflexion are uniquely determined projections of the points Ω through the points P upon the lines l. For each pencil, the imaginary points of inflexion are the projections of its pair Ω through the point P common to the three real harmonic polars upon the lines l which are sides of the real inflexional triangle; and the critic centers are the projections through the points P which are the vertices of the triangle upon the three real harmonic polars*. In all there are thirty-six critic centers, four real and thirty-two imaginary.

THEOREM VIII. *The complete set of critic centers and points of inflexion for the four pencils of cubics determined by four real points chosen arbitrarily in the plane as critic centers, form a set of fifty-four points, consisting of the four vertices of the chosen* 4*-point, the six vertices of the associated* 4*-line in the quadrangle-quadrilateral configuration, the two points on each side of the* 4*-line equianharmonic to the vertices, and thirty-six uniquely defined projections of the eight equianharmonic points upon the six sides of the* 4*-point through its vertices.*

7. **The second quadruple of pencils of cubics.** The three points D with each of the points P are the vertices of a 4-point, and the polar lines of the points P with respect to the triangle with the vertices D are the sides of the associated 4-line. Hence there are four pencils of cubics

* Annals of Math. loc. cit.

$$
\begin{aligned}
(\ x+y+z)(\ x+\omega y+\omega^2 z)(\ x+\omega^2 y+\omega z)+\lambda xyz &= 0,\\
(-x+y+z)(-x+\omega y+\omega^2 z)(-x+\omega^2 y+\omega z)-\lambda xyz &= 0,\\
(\ x-y+z)(\ x-\omega y+\omega^2 z)(\ x-\omega^2 y+\omega z)-\lambda xyz &= 0,\\
(\ x+y-z)(\ x+\omega y-\omega^2 z)(\ x+\omega^2 y-\omega z)-\lambda xyz &= 0;
\end{aligned}
$$

that is,

$$
\begin{aligned}
x^3+y^3+z^3+\mu xyz &= 0,\\
x^3-y^3-z^3+\mu xyz &= 0,\\
-x^3+y^3-z^3+\mu xyz &= 0,\\
-x^3-y^3+z^3+\mu xyz &= 0, \qquad [\mu = \lambda-3],
\end{aligned}
$$

with the points D as vertices of the real inflexional triangle, the points P as the remaining real critic centers, and the points I as real points of inflexion.

These four pencils of cubics have twelve imaginary points of inflexion

$$
\begin{array}{llll}
(\ 0, & 1, & \pm\omega), & (\ 0, \quad 1, \quad \pm\omega^2),\\
(\pm\omega, & 0, & 1\ ), & (\pm\omega^2, \quad 0, \quad 1\ ),\\
(\ 1, & \pm\omega, & 0\ ), & (\ 1, \quad \pm\omega^2, \quad 0\ ),
\end{array}
$$

each pair being the two points equianharmonic with respect to a point I and the two points D with which the point I is collinear. The four inflexional sets schematically arranged are

$$
\begin{array}{llllllllll}
(\ 0, & 1, & -1), & (\ 0, & 1, & -\omega), & (\ 0, & 1, & -\omega^2),\\
(-\omega^2, & 0, & 1\ ), & (-1, & 0, & 1\ ), & (-\omega, & 0, & 1\ ),\\
(\ 1, & -\omega, & 0\ ), & (\ 1, & -\omega^2, & 0\ ), & (\ 1, & -1, & 0\ );
\end{array}
$$

$$
\begin{array}{llllllllll}
(\ 0, & 1, & -1), & (\ 0, & 1, & -\omega), & (\ 0, & 1, & -\omega^2),\\
(\ \omega^2, & 0, & 1\ ), & (\ 1, & 0, & 1\ ), & (\ \omega, & 0, & 1\ ),\\
(\ 1, & \omega, & 0\ ), & (\ 1, & \omega^2, & 0\ ), & (\ 1, & 1, & 0\ );
\end{array}
$$

$$
\begin{array}{llllllllll}
(\ 0, & 1, & 1\ ), & (\ 0, & 1, & \omega\ ), & (\ 0, & 1, & \omega^2\ ),\\
(-\omega^2, & 0, & 1\ ), & (-1, & 0, & 1\ ), & (-\omega, & 0, & 1\ ),\\
(\ 1, & \omega, & 0\ ), & (\ 1, & \omega^2, & 0\ ), & (\ 1, & 1, & 0\ );
\end{array}
$$

$$
\begin{array}{llllllllll}
(\ 0, & 1, & 1\ ), & (\ 0, & 1, & \omega\ ), & (\ 0, & 1, & \omega^2\ ),\\
(\ \omega^2, & 0, & 1\ ), & 1, & 0, & 1\ ), & (\ \omega, & 0, & 1\ ),\\
(\ 1, & -\omega, & 0\ ), & (\ 1, & -\omega^2, & 0\ ), & (\ 1, & -1, & 0\ ).
\end{array}
$$

The complete set of inflexional axes are the sides of the diagonal triangle, the sides of the 4-line, the lines ψ_μ, and the twenty-four imaginary lines,

$$\begin{aligned} \pm x \pm y + \omega z &= 0, & \pm x \pm y + \omega^2 z &= 0, \\ \pm x + \omega y \pm z &= 0, & \pm x + \omega^2 y \pm z &= 0, \\ \omega x \pm y \pm z &= 0, & \omega^2 x \pm y \pm z &= 0. \end{aligned}$$

The remaining critic centers are

$$\begin{aligned} &(\pm 1, \pm 1, \omega), & &(\pm 1, \pm 1, \omega^2), \\ &(\pm 1, \omega, \pm 1), & &(\pm 1, \omega^2, \pm 1), \\ &(\omega, \pm 1, \pm 1), & &(\omega^2, \pm 1, \pm 1). \end{aligned}$$

The relations thus shown may be expressed in general in the following way.

THEOREM IX. *The seven vertices of a real* 4*-point and its diagonal triangle are the critic centers for four pencils of cubics with the diagonal triangle as real inflexional triangle: the sides of the* 4*-point are the real harmonic polars, and the six vertices of the associated* 4*-line in the quadrangle-quadrilateral configuration are the real points of inflexion.*

For the four pencils of cubics:

THEOREM X. *There are twelve imaginary points of inflexion, two pairs on each side of the diagonal triangle, each pair consisting of the two points equianharmonic to a vertex of the* 4*-line and the two vertices of the triangle with which it is collinear.*

THEOREM XI. *There is an inflexional set of six real and twelve imaginary points: each two of the four pencils have in common one real and two imaginary points of inflexion; and each of the eighteen points belongs to the inflexional sets for two, and only two, of the four pencils.*

THEOREM XII. *In addition to the three common real inflexional axes, there are thirty-six inflexional axes, four real and thirty-two imaginary; and the eighteen points of inflexion and thirty-six inflexional axes form a configuration* $(18_8\, 36_3)$.

When a triangle and one point are fixed, three other points are uniquely determined by harmonic properties such that the four points and triangle form a quadrangle and its diagonal triangle.

THEOREM XIII. *Every syzygetic pencil of cubics is associated with three other syzygetic pencils, such that the totality of critic centers and points of inflexion form a set of fifty-seven points: the vertices of a quadrangle-quadrilateral configuration and its diagonal triangle, the points equianharmonic*

to the vertices of the quadrilateral, and thirty-six uniquely defined projections of the equianharmonic points through the vertices of the triangle upon the sides of the quadrangle and through the vertices of the quadrangle upon the sides of the triangle.

8. **The configuration C_{13}.** In a paper, not yet published, by the writer it is shown that there are twelve, and only twelve, real pencils of cubics with five points of inflexion in common with a given real pencil; and the configuration composed of the original pencil and twelve such associated pencils is denoted by C_{13}. It is also shown that the thirteen such pencils of cubics have in all just forty points of inflexion, ten real and thirty imaginary; and that when the original pencil is taken as

$$x^3+y^3+z^3+\mu xyz = 0,$$

the forty points all lie on the thirteen real lines

$$y \pm z = 0, \quad z \pm x = 0, \quad x \pm y = 0,$$
$$x \pm y \pm z = 0,$$
$$x = 0, \quad y = 0, \quad z = 0.$$

As noted in this paper these thirteen lines form a quadrangle-quadrilateral configuration and the diagonal triangle common to the associated 4-point and 4-line. The thirty imaginary points were found to be intersections of the thirteen lines by the eight imaginary lines

$$\begin{aligned} x + y + \omega z &= 0, & x + y + \omega^2 z &= 0, \\ x + \omega y + z &= 0, & x + \omega^2 y + z &= 0, \\ \omega x + y + z &= 0, & \omega^2 x + y + z &= 0, \\ x + \omega y + \omega^2 z &= 0, & x + \omega^2 y + \omega z &= 0; \end{aligned}$$

and the ten real points to be

$$(0, 1, -1), \quad (-1, 0, 1), \quad (1, -1, 0), \quad (1, 1, 1),$$
$$(0, 1, 2), \quad (1, 0, 2), \quad (1, 2, 0),$$
$$(0, 2, 1), \quad (2, 0, 1), \quad (2, 1, 0).$$

It follows directly from the symmetrical relations, that the points of inflexion for the four configurations C_{13} originating from the four pencils of cubics

$$x^3+y^3+z^3+\mu xyz = 0,$$
$$x^3-y^3-z^3+\mu xyz = 0,$$
$$-x^3+y^3-z^3+\mu xyz = 0,$$
$$-x^3-y^3+z^3+\mu xyz = 0,$$

all lie on the thirteen real lines

$$y \pm z = 0, \quad z \pm x = 0, \quad x \pm y = 0,$$
$$x \pm y \pm z = 0,$$
$$x = 0, \quad y = 0, \quad z = 0.$$

The imaginary points are intersections of the thirteen real lines by the thirty-two imaginary lines

$$\pm x \pm y + \omega z = 0, \qquad \pm x \pm y + \omega^2 z = 0,$$
$$\pm x + \omega y \pm z = 0, \qquad \pm x + \omega^2 y \pm z = 0,$$
$$\omega x \pm y \pm z = 0, \qquad \omega^2 x \pm y \pm z = 0,$$
$$x \pm \omega y \pm \omega^2 z = 0, \qquad x \pm \omega^2 y \pm \omega z = 0;$$

and the real points are

$$(0, 1, \pm 1), \quad (\pm 1, 0, 1), \quad (1, \pm 1, 0), \quad (1, \pm 1, \pm 1),$$
$$(0, \pm 1, 2), \quad (2, 0, \pm 1), \quad (\pm 1, 2, 0),$$
$$(0, 2, \pm 1), \quad (\pm 1, 0, 2), \quad (2, \pm 1, 0).$$

It is known that the fifteen diagonal lines of the 10_3 are of two kinds: those of one kind being the three sides of the diagonal triangle common to the associated 4-point and 4-line; and those of the other kind being twelve joins,

$$\pm x \pm y + 2z = 0, \quad \pm x + 2y \pm z = 0, \quad 2x \pm y \pm z = 0,$$

of the points P with the points I. The points

$$(0, 1, \pm 1), \quad (\pm 1, 0, 1), \quad (1, \pm 1, 0), \quad (1, \pm 1, \pm 1)$$

are the vertices of tho 10_3; and the points

$$(0, \pm 1, 2), \quad (2, 0, \pm 1), \quad (\pm 1, 2, 0),$$
$$(0, 2, \pm 1), \quad (\pm 1, 0, 2), \quad (2, \pm 1, 0)$$

are the intersections of

$$x = 0, \quad y = 0, \quad z = 0$$

with

$$\pm x \pm y + 2z = 0, \quad \pm x + 2y \pm z = 0, \quad 2x \pm y \pm z = 0.$$

THEOREM XIV. *The points of inflexion for the four C_{13}'s which have as origin four pencils of cubics with the diagonal triangle common to the associated 4-point and 4-line in a quadrangle-quadrilateral configuration as real inflexional triangle and the remaining real critic centers at the vertices of the 4-point, all lie on the sides of the quadrangle-quadrilateral configuration: the imaginary points of inflexion are intersections of the 10_3 by the imaginary inflexional axes of the four pencils; the real points of inflexion are the vertices of the 10_3 and the intersections of the three diagonal lines of one kind by the twelve diagonal lines of the second kind.*

The twelve points

$$(0, \pm 1, 2), \quad (2, 0, \pm 1), \quad (\pm 1, 2, 0),$$
$$(0, 2, \pm 1), \quad (\pm 1, 0, 2), \quad (2, \pm 1, 0)$$

lie by sixes on four conics,

$$2(x^2+y^2+z^2) - 5(yz+zx+xy) = 0,$$
$$2(x^2+y^2+z^2) - 5(yz-zx-xy) = 0,$$
$$2(x^2+y^2+z^2) - 5(-yz+zx-xy) = 0,$$
$$2(x^2+y^2+z^2) - 5(-yz-zx+xy) = 0,$$

which have double contact with the conic K at the points Ω with the lines Ψ as the common tangents.

It is also known that the diagonal points of the 10_3 are of two kinds, the three vertices D of the diagonal triangle common to the 4-point and 4-line, and twelve meets J,

$$(\pm 1, \pm 1, 2), \quad (\pm 1, 2, \pm 1), \quad (2, \pm 1, \pm 1),$$

of the six sides of the 4-point with the sides of the 4-line. Investigation shows that the set of real points of inflexion for the four configurations C_{13}, which have as origin the four pencils of cubics in the first quadruple considered, are the vertices of the 10_3 and the points J. Also the imaginary

points of inflexion for these four configurations C_{12} all lie on the configuration or diagonal lines of the 10_3.

THEOREM XV. *The points of inflexion for the four C_{12}'s, which have as origin the pencils of cubics determined by the vertices of a real 4-point as critic centers, all lie on the configuration lines or diagonal lines of the quadrangle-quadrilateral configuration determined by the 4-point: the real points of the set are the ten vertices of the 10_3 and the twelve diagonal points which are meets of the sides of the 4-point with the sides of the 4-line.*

The twelve points J also lie by sixes on four conics

$$\begin{aligned} x^2+y^2+z^2+6(\ \ yz+zx+xy) &= 0,\\ x^2+y^2+z^2+6(\ -yz-zx-xy) &= 0,\\ x^2+y^2+z^2+6(-yz+zx-xy) &= 0,\\ x^2+y^2+z^2+6(-yz-zx+xy) &= 0, \end{aligned}$$

which have double contact with the conic K at the points Ω with the lines Ψ as the common tangents.

9. **Covariants to four points.** It is well known that four independent points in a plane are permuted in all possible ways by Klein's group G_{41} of twenty-four collineations simply isomorphic to the symmetric group on four letters*; and that the transformations of the group permute in the same order the corresponding lines in the associated 4-line. It follows directly from Theorem I, that the permutation of the four points is accompanied by a similar permutation of the four pencils of cubics in the first quadruple considered.

The cubics of the second quadruple have a common real inflexional triangle, and the four pencils are permuted as the fourth real critic center is carried from one to the other of the vertices of the 4-point by the G_{41}.

From their relative positions it follows that the permutation of the pencils of cubics in each of the quadruples is accompanied by a similar permutation of the four associated configurations C_{12}.

Generators of the group G_{41} for the points P†, and hence for the sets of cubics considered are

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* Klein: Über eine Geometrische Repräsentation der Resolventen Algebraischer Gleichungen; Math. Annalen, vol. 4 (1871), p. 346.

Moore: Concerning Klein's Group of $(n+1)!$ n-ary Collineations; Amer. Jour. of Math., vol. 22 (1900), p. 336.

Morley: Projective Coordinates; Trans. Amer. Math. Soc., vol. 4 (1903), p. 291.

† Winger: Projective Geometry, p. 345.

$$\begin{array}{lcccc} & R_1 & R_2 & S & T \\ x' = & -x & x & z & x \\ y' = & y & -y & x & z \\ z' = & z & z & y & y. \end{array}$$

THEOREM XVI. *Covariant to four points under Klein's group $G_{4!}$ are two quadruples of pencils of cubics, and two sets of four configurations C_{12}.*

As known, in the quadrangle-quadrilateral configuration the elements of the 4-point and the elements of the 4-line are not equivalent under the operations of the group. It follows, from their association with the points and lines of the 10_3, that the elements of the larger configurations discussed in this paper are separated into two kinds in accordance with their respective relations to the 4-point and 4-line; and that consequently the configurations are not regular, that is, not transitive.

UNIVERSITY OF WEST VIRGINIA,
August, 1923.

ON PELLET'S THEOREM CONCERNING THE ROOTS OF A POLYNOMIAL.*

By J. L. Walsh.

1. Introduction. It is the object of this note to present a converse and to indicate some applications of the following theorem due to Pellet:†

If the polynomial

$$F(x) \equiv |a_0| + |a_1|x + |a_2|x^2 + \cdots + |a_{n-1}|x^{n-1} - |a_n|x^n + |a_{n+1}|x^{n+1} + |a_{n+2}|x^{n+2} + \cdots + |a_k|x^k \tag{1}$$

has two positive roots x_1 and x_2 $(x_1 < x_2)$, then the polynomial

$$f(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \tag{2}$$

has no roots in the annular region $x_1 < |x| < x_2$, and has precisely n roots whose absolute value is not greater than x_1.

Pellet's proof depends on the following easily proved theorem due to Rouché:‡

If two functions $f_1(x)$ and $f_2(x)$ are analytic on and within a closed curve, and if on this curve we have

$$|f_1(x)| > |f_2(x)|$$

then the two equations

$$f_1(x) = 0, \qquad f_1(x) + f_2(x) = 0,$$

have the same number of roots interior to the curve.

2. Proof of Pellet's Theorem. Pellet's proof of his theorem is quite simple, but we give a new§ proof which is even more elementary, and which is more in keeping with the methods we shall use later.

The polynomial $F(x)$ has, by Descartes' rule of signs, either no positive root or precisely two positive roots, so if the hypothesis of the theorem is satisfied, there are no positive roots other than x_1 and x_2. We assume,

* Presented to the American Mathematical Society, Dec. 1923.

† A. E. Pellet, Darboux's Bulletin (2), vol. 5 (1881), pp. 393-395.

‡ Journal de l'École Polytechnique, vol. 22 (1862), pp. 217-218.

§ See, however, Picard, Nouv. Ann. (3), vol. 11 (1892), pp. 147-148 and the reference there given to Mayer.

as we may do without loss of generality, that neither a_0 nor a_k vanishes. Then $F(0)$ and $F(+\infty)$ are both positive, so $F(x)$ is negative if $x_1 < x < x_2$:

$$(3) \qquad \begin{aligned} |a_n|x^n > |a_0| + |a_1|x + \cdots + |a_{n-1}|x^{n-1} \\ + |a_{n+1}|x^{n+1} + |a_{n+2}|x^{n+2} + \cdots + |a_k|x^k. \end{aligned}$$

If we do not restrict x to being positive, but require merely that x shall lie in the annular region $x_1 < |x| < x_2$, we have by (3),

$$(4) \qquad \begin{aligned} |a_n|\,|x|^n > |a_0| + |a_1|\,|x| + \cdots + |a_{n-1}|\,|x|^{n-1} \\ + |a_{n+1}|\,|x|^{n+1} + |a_{n+2}|\,|x|^{n+2} + \cdots + |a_k|\,|x|^k, \end{aligned}$$

and thus $f(x)$ cannot vanish.

It is clearly true that $f(x)$ cannot vanish in the annular region so long as (3) is satisfied, even if the coefficients a_i are allowed to vary. Let us allow $|a_0|, |a_1|, \cdots, |a_{n-1}|$ to vary continuously, and monotonically to approach zero. This variation simply strengthens the inequality (3), and x_1 (for which (3) becomes an *equality*) decreases monotonically. We suppose that the coefficients $a_n, a_{n+1}, \cdots, a_k$ are kept fixed. The root x_2 increases monotonically. However, x_1 approaches zero, together with n roots of $f(x)$. The variation of the roots of $f(x)$ is continuous, no root x ever lies in the annular region $x_1 < |x| < x_2$, so there are always n roots x of $f(x)$ in the circle $|x| \leqq x_1$. The proof of Pellet's Theorem is now complete.

Cauchy established* a result that is essentially a special case of Pellet's Theorem, namely that all roots of $f(x)$ are in absolute value less than or equal to the positive root x_1 of the equation

$$(5) \qquad |a_0| + |a_1|x + \cdots + |a_{k-1}|x^{k-1} - |a_k|x^k = 0;$$

we set $x_1 = \infty$ if $a_k = 0$. We find by the substitution $x = 1/x'$ and application of Cauchy's result that all roots of $f(x)$ are in absolute value greater than or equal to the positive (or zero) root of the equation

$$(6) \qquad |a_0| - |a_1|x - |a_2|x^2 - \cdots - |a_k|x^k = 0.$$

It is a direct corollary to Pellet's Theorem that if (3) is satisfied for a particular real value of x, then that value of $|x|$ separates the n roots of $f(x)$ of smallest moduli from the remaining $n-k$ roots.

* Exercises de Mathématiques, 1829; Œuvres (2), vol. 9, p. 122.

The proof of Pellet's Theorem holds with only minor changes for a function $f(x)$ which is an entire function or a convergent power series.

3. **A converse of Pellet's Theorem.** We shall add to Pellet's Theorem a converse, to the effect that *the theorem (including the results due to Cauchy) gives all possible regions which contain no roots of $f(x)$ but whose determination depends only on the absolute values of the coefficients* $a_0, a_1, \cdots, a_k$.

We assume $a_k \neq 0$, which involves no loss of generality. We shall suppose, then, that the absolute values of the coefficients of $f(x)$ are given, and we are to determine the locus of the roots of $f(x)$ as the coefficients vary independently in all possible ways while having the prescribed moduli.

The points Q of the locus of the roots of $f(x)$ do not comprise the entire plane, by Cauchy's result. The point set composed of all possible points Q is closed, for the roots of $f(x)$ are continuous functions of the coefficients. This point set has circular symmetry about the origin, for if x is a root of $f(x)$, the substitution $x' = \omega x$, where $|\omega| = 1$, does not alter the absolute values of the coefficients of (2) yet can be chosen so as to transform x into any preassigned point x' equidistant with x from the origin.

The limit given by Cauchy is the best possible upper limit for the absolute values of the roots which depends only on the absolute values of the coefficients. For that limit is an actual root of a polynomial of type (2) whose coefficients have the prescribed absolute values. Similarly the lower limit defined by (6) is the best possible lower limit for the absolute values of the roots which depends merely on the absolute values of the coefficients.

Let us now suppose that there is no polynomial $f(x)$ whose coefficients have the prescribed absolute values and which has a root of modulus a; we shall prove that (3) is satisfied for $x = a$ for some choice of n. The number of roots of $f(x)$ whose modulus is less than a does not depend on the particular equation $f(x)$ chosen, the coefficients restricted as prescribed, for any two such equations can be changed one into the other by a continuous change of the coefficients (always satisfying the prescribed condition), the roots then vary continuously, and no root can cross the circle $|x| = a$.

If there are n roots of $f(x)$ whose moduli are less than a, let us consider the expression,

$$(7) \qquad \begin{aligned} -F(a) = {} & |a_n| a^n - |a_0| - |a_1| a - \cdots - |a_{n-1}| a^{n-1} \\ & - |a_{n+1}| a^{n+1} - |a_{n+2}| a^{n+2} - \cdots - |a_k| a^k; \end{aligned}$$

we shall prove it to be positive. Allow the n roots of $f(x)$ which are in absolute value less than a to vary continuously and monotonically (in

absolute value), and to approach zero; this variation alters the coefficients of $f(x)$ but does not alter the remaining $k-n$ roots of $f(x)$ whose absolute values are greater than a. During the variation as described, the expression (7) never vanishes, and at the end of the process (7) is positive, by the test connected with (6). Thus (7) is positive throughout the process, (3) is satisfied for $x = a$, $F(x)$ has two distinct positive roots x_1 and x_2, we have $x_1 < a < x_2$, and Pellet's Theorem gives an annulus containing the circle $|x| = a$ such that $f(x)$ has no root in the annulus.

The converse of Pellet's Theorem is now completely established.

4. A limiting case of Pellet's Theorem. It is interesting to consider a limiting case of Pellet's Theorem, namely the case that $F(x)$ has one double positive root x_1; we have

$$(8) \qquad \begin{aligned} &|a_0| + |a_1| x_1 + \cdots + |a_{n-1}| x_1^{n-1} \\ &\qquad - |a_n| x_1^n + |a_{n+1}| x_1^{n+1} + \cdots + |a_k| x_1^k = 0. \end{aligned}$$

Of course $F(x)$ has no other positive root. We shall find it convenient to suppose $a_0 > 0$, which involves no loss in generality. Let us determine the circumstances under which $f(x)$ has roots with the modulus x_1. We must have

$$(9) \qquad \begin{aligned} -a_n x^n = a_0 &+ a_1 x + \cdots + a_{n-1} x^{n-1} \\ &+ a_{n+1} x^{n+1} + \cdots + a_k x^k. \end{aligned}$$

When we notice (8), the equation $|x| = x_1$, and the inequality $a_0 > 0$, the following equations are seen to be necessary and sufficient in order that (9) should obtain:

$$\begin{aligned} a_1 x &= |a_1| x_1, & a_{n+1} x^{n+1} &= |a_{n+1}| x_1^{n+1}, \\ a_2 x^2 &= |a_2| x_1^2, & a_{n+2} x^{n+2} &= |a_{n+2}| x_1^{n+2}, \\ &\cdots\cdots\cdots, & &\cdots\cdots\cdots, \\ a_{n-1} x^{n-1} &= |a_{n-1}| x_1^{n-1}, & a_k x^k &= |a_k| x_1^k. \\ a_n x^n &= -|a_n| x_1^n, \end{aligned}$$

The substitution $\omega x = x_1$ (where $|\omega| = 1$) reduces these equations to

$$
\begin{aligned}
a_1 &= |a_1|\omega, & a_n &= -|a_n|\omega^n,\\
a_2 &= |a_2|\omega^2, & a_{n+1} &= |a_{n+1}|\omega^{n+1},\\
&\cdots\cdots\cdots, & &\cdots\cdots\cdots,\\
a_{n-1} &= |a_{n-1}|\omega^{n-1}, & a_k &= |a_k|\omega^k.
\end{aligned}
\tag{10}
$$

These equations are a necessary and sufficient condition that $f(x)$ should have one or more roots whose modulus is x_1. We shall now determine the precise number of such roots.

If equations (10) are satisfied by a number ω, a necessary and sufficient condition that (10) should be satisfied by ω_1 is the system

$$
\omega^{m_1} = \omega_1^{m_1}, \quad \omega^{m_2} = \omega_1^{m_2}, \quad \cdots, \quad \omega^{m_\nu} = \omega_1^{m_\nu}, \tag{11}
$$

where $a_{m_1}, a_{m_2}, \cdots, a_{m_\nu}$ are those numbers a_i which are different from zero. If δ is the greatest common divisor of $m_1, m_2, \cdots, m_\nu$, a necessary and sufficient condition for equations (11) is the single equation

$$
\omega^\delta = \omega_1^\delta.
$$

That is, there are precisely δ values of ω which lead to a root of $f(x)$,* and each of these is a double root of $f(x)$, since that polynomial can be transformed into $F(x)$ by a rotation of the plane.

If equations (10) have at least one solution, let us vary $a_0, a_1, \cdots, a_{n-1}$ by decreasing monotonically their moduli and by keeping fixed their arguments. Inequality (4) for $x = x_1$ thereby replaces (8), so $F(x)$ has two positive roots, caused by the separation of the two roots which coincided at x_1; these two roots become respectively greater and less than x_1. Equations (10) are not affected by the process, so the roots of $f(x)$ whose moduli are x_1 divide, and each double root separates into two distinct roots whose moduli are respectively greater and less than x_1. Thus we have, with the aid of Pellet's Theorem,

If $F(x)$ has two coincident positive roots at x_1, then either $f(x)$ has precisely n roots whose moduli are less than x_1 and precisely $k - n$ roots

* The reasoning thus far applies whether $F(x)$ has one positive root or two positive roots.

whose moduli are greater than x_1, or $f(x)$ has precisely δ double roots whose moduli are equal to x_1, precisely $n - \delta$ roots whose moduli are less than x_1, and precisely $k - n - \delta$ roots whose moduli are greater than x_1.

In the latter case, of course, only powers of x^δ appear in $f(x)$.

5. **Applications of Pellet's Theorem.** A number of applications of Pellet's Theorem have been made in the literature. The following application is not difficult to make:*

An equation of $k+2$ terms whose terms of least degree are 1 and x^p has always p roots whose moduli are not greater than a fixed number $\mu(k)$ which depends only on k.

This result was proved for the case $p = 1$, $k = 1, 2$ by Landau,† using Pellet's Theorem. The result was proved for $p = 1$ and all values of k by Fejér,‡ although by a method not involving Pellet's Theorem.

The following theorem is a generalization of these results, and can be established by the method indicated in the note by the author to which reference has just been made:

There exists a number $\mu(a, k)$ such that every polynomial

$$1 + a_1 x^{\nu_1} + a_2 x^{\nu_2} + \cdots + a_{h-1} x^{\nu_{h-1}} + a x^{\nu_h} + a_{h+1} x^{\nu_{h+1}} + a_{h+2} x^{\nu_{h+2}} + \cdots + a_k x^{\nu_k}$$

has at least ν_1 roots whose moduli are not greater than $\mu(a, k)$, no matter what may be the numbers h, $a_1, a_2, \cdots, a_{h-1}, a_{h+1}, \cdots, a_k$, $\nu_1, \nu_2, \cdots, \nu_k$, provided merely that

$$0 < \nu_1 < \nu_2 < \cdots < \nu_k.$$

We add the remark that Pellet's Theorem seems to have important applications to the theory of Graeffe's method for the computation of the roots of an algebraic equation. Those applications have not yet been made in detail.§

* Walsh, Comptes Rendus, vol. 176 (1923), pp. 1361–1363. See also Biernacki, Comptes Rendus, vol. 177 (1923), pp. 1193–1194.

† Annales de l'École Normale (3), vol. 24 (1907), pp. 198–201.

‡ Math. Annalen, vol. 65 (1907–08), pp. 413–423.

§ See, however, M. E. Carvallo, *Méthode pratique pour la résolution numérique complète des équations algébriques ou transcendantes* (Paris, 1896).

THE SETTING OF A PROPOSITION.

By P. J. Daniell.

1. Introduction.

1.1. When we consider an ordinary proposition, such as "Napoleon died on St. Helena" or "the sun is hot", from the point of view of the Principia Mathematica, we observe immediately that it contains descriptive elements. It is not a strict proposition unless we regard it as enlarged by the consideration of the properties (propositional functions) which determine the entities described. A full analysis of "the sun is hot" would include the descriptive functions which give meaning to the term "the sun" if not also to "x is hot". The point of view of this paper is that this determination is not to be regarded as a tacitly understood part of the proposition itself, but is given by the setting or context in which the proposition appears. Even the Principia Mathematica* admits that the symbol for a proposition is incomplete.

Our view is that all "propositions" which are communicable from one person to another by means of symbols must have a certain degree of incompleteness; that such fragments of rational thought have the nature of propositional functions; that strict propositions are abstract ideal limiting elements like points in geometry or else are functions which are true always or true never. Certainly a "proposition" which can even be imagined to be false in one context, true in another must be regarded as a propositional function if we are to follow the methods of the Principia Mathematica.

1.2. It is an essential part of a rational system expressed symbolically (that is, in any form of language) that certain terms appear again and again. Frequently the exact meaning of a particular term or symbol only becomes definite through such a set of propositions, or, as we prefer, rational fragments, about it. In other words, in such a system we may consider two propositional functions fx, gx in which the variable x is to be the same in both. There thus arises the question of mutual relevance of such rational fragments.

1.3. The setting consists of a collection P of such fragments. A particular fragment, f, such as "the sun is hot" will be relevant in the setting if P includes statements about the sun and statements including "is hot".

Furthermore we may have both f and not-f compatible with P. In this there may appear to be some similarity to the ideas of C. I. Lewis† who

* Russel and Whitehead, Principia Mathematica, p. 46.

† C. I. Lewis, Survey of Symbolic Logic, Chap. 5.

makes a distinction between the false and the logically impossible. But we agree with Russell and Whitehead that there is no such distinction for strict propositions. To regard a proposition as false accidentally is to regard it as a rational fragment or propositional function which is not false for all its values but excluded from a particular setting.

1.4. Since the notion of relevance is not a self-evident property of propositional functions as they are handled in the Principia Mathematica, we prefer to regard our "rational fragments of thought-processes", denoted by $f, g, h, \cdots$ as distinct from propositional functions and to handle them by means of new primitive assertions about them.

1.5. We denote our rational fragments by $f, \cdots$ and we suppose that there is a negative of f denoted by $-f$, and a logical product fg. Moreover there may be a relation "f implies g" denoted by

$$f < g$$

which has the nature of a formal implication between propositional functions, so that g belongs logically to, and not-g is excluded from, any rational system to which f belongs.

If the symbols in the expression of f refer to entities of which all appear in g we say that f is relevant with respect to g and denote it by

$$f \text{ rel } g.$$

For formal reasons we regard relevance as a primitive concept. Again there is the part implied by f which is relevant with respect to g, denoted by

$$f((g)).$$

This is also taken to be primitive. It is possible to regard it as an incomplete symbol so that statements about it are really about f and g, but that would increase the length of this paper.

Propositions of the types $f<g$ and $f \text{ rel } g$ are regarded as strict propositions in the sense of the Principia Mathematica and there may be relations of implication between some of them. Such implications (in the sense of the Principia Mathematica) are expressed by the word "implies".

2. Primitive assertions and definitions.

2.11. $-(-f)<f$. 2.12. $f<-(-f)$.

2.13. $fg<f$. 2.14. $fg<g$.

2.15. $(f<g)(g<h)$ implies $(f<h)$.

2.16. $(f<g)(f<h)$ implies $(f<gh)$.

2.17. $fg<-h$ implies $fh<-g$.

2.21. $-f$ rel f. 2.22. f rel $-f$.
2.23. f rel fg. 2.24. g rel fg.
2.25. $(f$ rel $g)\,(g$ rel $h)$ implies $(f$ rel $h)$.
2.26. $(f$ rel $h)\,(g$ rel $h)$ implies $(fg$ rel $h)$.
2.31. $f<f((g))$.
2.32. $(f<g)\,(g$ rel $h)$ implies $f((h))<g$.
2.33. $f((g))$ rel g.
2.34. g rel h implies g rel $f((h))$.

[When g rel h, $f((h))$ includes $g+(-g)$ as a factor.]

2.41. $f|g$ denotes $f<-g$.
2.42. $f=g$ denotes $(f<g)\,(g<f)$.
2.43. $f=0$ denotes $f<-f$.
2.44. $f=1$ denotes $-f<f$.
2.45. $f+g$ denotes $-(-f)\,(-g)$.
2.51. $f.<r.g$ denotes $(f<g)\,(g$ rel $f)$.
2.52. $f.=r.g$ denotes $(f.<r.g)\,(g.<r.f)$.

The symbol in 2.51 can be read "f relevantly implies g" and that in 2.52 as "f is relevantly equivalent to g".

3. **Immediate consequences.**

3.11. $f<f$, from 2.12, 2.11 and 2.15.
3.12. $f=f$.
3.13. $f<ff$, from 3.11 twice and 2.16.
3.14. $f=ff$. 3.15. $f=-(-f)$.
3.16. $fg<gf$, from 2.14, 2.13 and 2.16.
3.17. $fg=gf$.
3.18. f rel f, from 2.22, 2.21 and 2.25.
3.19. f rel $-(-f)$, from 2.22 as it stands and then with $-f$ in place of f, and 2.25.
3.20. $-(-f)$ rel f.
3.21. $-(-f).<r.f$. 3.22. $f.<r.-(-f)$.
3.23. $fg.<r.f$. 3.24. $fg.<r.gf$.
3.25. $(f.<r.g)\,(g.<r.h)$ implies $(f.<r.h)$.
3.26. $(f.<r.g)\,(f.<r.h)$ implies $(f.<r.gh)$.

The theorems 3.21 to 3.26 have the same forms as 2.11 to 2.16 with $.<r.$ in place of $<$, but 2.17 has no such analogy.

3.27. $f.<r.f$. 3.28. $f.=r.f$.
3.29. $f.<r.ff$. 3.30. $f.=r.ff$.
3.31. $f.=r.-(-f)$. 3.32. $fg.=r.gf$.

When p, q are strict propositions, having asserted p we have the right to assert

q implies pq.

This method will be used frequently tn the following theorems.

3.33. $f<h$ implies $fg<h$.

In this case $fg<f$ takes the place of p. Then $f<h$ implies $(fg<f)(f<h)$, implies $fg<h$.

3.34. $f<h$ implies $fg<gh$.

For $f<h$ implies $(fg<g)(fg<h)$.

3.35. $(f<g)(h<k)$ implies $fh<gk$.

For

$(f<g)(h<k)$ implies $f<g$, implies $fh<hg$.

$(f<g)(h<k)$ implies $h<k$, implies $hg<gk$.

3.36. $fg<h$ implies $fg<fh$.

Since we have made no use of 2.17 theorems 3.33 to 3.36 remain true with $.<r.$ in place of $<$.

3.37. $f\ .<r.h$ implies $fg.<r.h$.

3.38. $f\ .<r.h$ implies $fg.<r.gh$.

3.39. $(f.<r.g)(h.<r.k)$ implies $fh.<r.gk$.

3.40. $fg.<r.h$ implies $fg.<r.fh$.

3.41. $f<g$ implies $-g<-f$.

For $f<g$

implies $f(-g)<g$ and $g<-(-g)$, by 3.33, 2.12,

implies $(-g)f<-(-g)$,

implies $(-g)(-g)<-f$, by 2.17,

implies $-g<-f$.

3.411. $f=g$ implies $-f=-g$.

3.42. $f=g$ implies $fh=gh$.

3.43. $(f=g)(g<h)$ implies $f<h$.

3.44. $(f=g)(h<g)$ implies $h<f$.

3.45. $(f=g)(g=0)$ implies $f=0$.

3.46. $(f=g)(g=1)$ implies $f=1$.

3.47. $(f=g)\ (g\,|\,h)$ implies $f\,|\,h$.

3.48. $(f=g)\ (h\,|\,g)$ implies $h\,|\,f$.

3.49. $(f=g)(g=h)$ implies $f=h$.

Hence when $f=g$ we are justified in substituting f for g provided relevance is not being considered.

3.50. f rel g implies f rel $-g$.

3.51. f rel g implies $-f$ rel g.

3.52. f rel g implies $-f$ rel $-g$.

3.53. $f\ .=r.g$ implies $-f.=r.-g$.

3.54. $f\ .=r.g$ implies $fh.=r.gh$.

3.55. $(f.=r.g)(g$ rel $h)$ implies f rel h.

3.56. $(f.=r.g)(h$ rel $g)$ implies h rel f.

These, together with 3.41 to 3.49 show that substitution of f for g is legitimate in any assertion when $f.=r.g$.

3.57. $-g<-(fg)$, from 3.41.

3.58. $f|g$ implies $fg=0$.

For, by 3.57, $f<-g$ implies $fg<-(fg)$.

3.59. $f(gh).<r.(fg)h$.

For

$f(gh).<r.gh,.<r.g. \qquad f(gh).<r.h.$

$f(gh).<r.f.$

$f(gh).<r.fg. \qquad f(gh).<r.h.$

3.60. $f(gh).=r.(fg)h$.

3.61. $fg=0$ implies $g<-f$.

For by 2.17 and the use of equivalent fragments,

$fg<-(fg)$ implies $f(fg)<-g$,
implies $fg<-g$,
implies $gf<-g$,
implies $gg<-f$.

3.62. $f|g$ implies $g|f$. 3.63. $f|g$ is equivalent to $g|f$.

3.64. $(f=0)(g<f)$ implies $g=0$.

For

$(f=0)(g<f)$ implies $(g<-f)(-f<-g)$,

3.65. $f=0$ implies $fg=0$.

3.66. $f<g$ is equivalent to $f(-g)=0$.

3.67. $f=0$ implies $f<g$.

3.68. $(f=0)(g=0)$ implies $f=g$,

This justifies the notation $f=0$.

3.70. $f<f+g$.

For

$(-f)(-g)<-f$.

$f<-(-f)<-(-f)(-g)$.

3.71. $(f<h)(g<h)$ implies $f+g<h$.

For

$(f<h)(g<h)$ implies $(-h<-f)(-h<-g)$,
implies $[-h<(-f)(-g)]$,
implies $-(-f)(-g)<-(-h)<h$.

3.72. fg rel $f+g$.

For, using 2.26, 3.50,

f rel $(-f)$ rel $(-f)(-g)$, g rel $(-f)(-g)$.

fg rel $-(-f)(-g)$.

3.73. $f+g$ rel fg.

This is proved similarly by means of 3.51.

3.74. $f+g \,.=\, r\,.\,g+f.$

3.75. $(f \text{ rel } h)\,(g \text{ rel } h)$ implies $(f+g \text{ rel } h)$.

3.76. $(f\,.<r.\,h)\,(g\,.<r.\,h)$ implies $(f+g\,.<r\,.\,h)$.

3.77. $fg+fh<f(g+h)$.

For by 3.71,

$$fg+fh<f,\quad fg<g<g+h,\quad fh<g+h,$$
$$fg+fh<g+h.$$

3.78. $fg+fh\,.<r\,.f(g+h)$.

For, since

$$(fg)(fh)\,.\,=\,r\,.\,[(fg)f]h\,.\,=\,r\,.\,(fg)h\,.\,=\,r\,.f(gh),$$
$$f(g+h) \text{ rel } f(gh) \text{ rel } (fg)(fh) \text{ rel } fg+fh.$$

3.79. $f(g+h)<fg+fh$.

We have in succession,

$$fg\,|-(fg),\quad f[-(fg)]\,|\,g,$$
$$f[-(fg)]\,[-(fh)]<f[-(fg)]<-g,$$
$$f[-(fg)]\,[-(fh)]<(-g)\,(-h)\,=\,-(g+h),$$
$$f(g+h)<-[-(fg)]\,[-(fh)].$$

3.80. $f(g+h)\,.<r\,.fg+fh.$

3.81. $f(g+h)\,.=r\,.fg+fh.$

3.82. $f+gh\;\;.=r\,.\,(f+g)\,(f+h)$, from 3.81 taking negatives.

3.83. $-(fg)\;\;.=r\,.\,(-f)+(-g).$

3.84. $f=g$ implies $f+h=g+h$.

3.85. $f.=r\,.g$ implies $f+h\,.=r\,.g+h.$

3.86. $f=1$ is equivalent to $-f=0$.

3.87. $(f=1)\,(g=1)$ implies $f=g$.

3.88. $(f=1)\,(f<g)$ implies $g=1$.

3.89. $f=1$ implies $f+g=1$.

3.90. $f<g$ is equivalent to $-f+g=1$.

3.91. $f=1$ implies $g<f$.

3.92. $f(-f)=0.$ 3.93. $f+(-f)=1.$

3.94. $(f=0)\,(g=0)$ is equivalent to $f+g=0$.

For

$$(f<-f)\,(g<-g) \text{ implies } [f<(-f)\,(-g)]\,[g<(-f)\,(-g)]$$
$$\text{implies } f+g<(-f)\,(-g)\,=\,-(f+g).$$

Since

$$f<f+g,\ f+g=0 \text{ implies } f=0 \text{ and } g=0.$$

3.95. $f=g$ is equivalent to $f(-g)+g(-f)=0$.

3.96. $f=0$ implies $f+g=g$.

Use 3.94 and 3.95.

3.97. $f=1$ implies $fg=g$.

3.98. $f[g+(-g)]=f=fg+f(-g).$

In all this we find that many of the axioms of traditional symbolic logic are satisfied with $.=r.$ in place of $=$. Thus

$$f+g.=r.g+f, \qquad fg.=r.gf,$$
$$(f+g)+h.=r.f+(g+h), \qquad (fg)h.=r.f(gh),$$
$$f(g+h).=r.fg+fh.$$

But if we give any meaning to $f.=r.0$ it must be equivalent to $f=0$, and

$(f=0)(g=0)$ does not imply $f.=r.g$.

$f.<r.g$ does not imply $-g.<r.-f$,

and it is not true, in general, that $f.<r.f+g$.

4. **Particular settings.** In a particular setting we regard every proposition (rational fragment) as tacitly multiplied by the propositional part of the setting P. By $-f$ we no longer mean the general not-f of universal logic but the combination $P(-f)$. Moreover we have the following theorems.

4.11. $P[-(-f)]<Pf$. 4.12. $Pf<P[-(-f)]$.

4.13. $P(fg)<Pf$. 4.14. $P(fg)<Pg$.

4.15. $(Pf<Pg)\ (Pg<Ph)$ implies $Pf<Ph$.

4.16. $(Pf<Pg)\ (Pf<Ph)$ implies $Pf<P(gh)$.

4.17. $P(fg)<P(-h)$ implies $P(fh)<P(-g)$.

4.21. $P(-f)$ rel Pf. 4.22. Pf rel $P(-f)$.

4.23. Pf rel $P(fg)$. 4.24. Pg rel $P(fg)$.

4.25. $(Pf$ rel $Pg)\ (Pg$ rel $Ph)$ implies Pf rel Ph.

4.26. $(Pf$ rel $Ph)\ (Pg$ rel $Ph)$ implies $P(fg)$ rel Ph.

4.43. In a particular setting, P, a fragment, f, is null when P excludes f, that is, when

$$Pf=0.$$

4.44. Similarly f is necessary in P when

$P<f$, and this is equivalent to $Pf=P$.

Thus we have in a particular setting the same logic as for the universal setting with $P0$ replaced by 0 and $P1$ by P and all the consequences in 3 remain true with every proposition multiplied tacitly by P. [We have made no use as yet of 2.31 to 2.34].

The following definitions are useful.

4.46. $f\gamma g$ denotes $-(f|g)$.

That is, f is compatiple with g means that f does not exclude g.

4.51. $f.<P.g$ denotes $(f\gamma P)\ (-g\gamma P)\ (Pf.<r.g)$.

That is, f dynamically implies g in the setting P when f is compatiple with P, P does not imply g, and Pf relevantly implies g.

We then obtain the following theorems.

4.62. $(f.<P.g)\ (g.<P.h)$ implies $f.<P.h$.

4.63. $(f.<P.g)\ (f.<P.h)$ implies $f.<P.gh$.

For

$$-(P<g) \text{ implies } -(P<gh).$$

4.64. $(f \text{ rel } Pg)\ (f.<P.g)$ implies $-g.<P.-f.$

4.65. $(fh\gamma P)\ (f.<P.g)\ (h.<P.k)$ implies $fh.<P.gk.$

4.66. $fg.<P.h$ implies $fg.<P.fh.$

This dynamic implication is a very narrow relation and does not hold when f is null relative to the setting. For most purposes the more useful relation would be the broader one

$$fP.<r.g,$$

but the narrrower relation comes nearer to one of the common meanings assigned to the term implication.

We add a few consequences of the definition of compatibility.

4.71. $f\gamma g$ implies $f\gamma(g+h)$.

For

$$(g+h)|f \text{ implies } fg+fh=0,$$
$$\text{implies } fg=0, \text{ that is, } g|f.$$

4.72. $f\gamma g$ is equivalent to $g\gamma f$.

4.73. $-(f=0)$ is equivalent to $f\gamma f$.

4.74. $[-(f=0)]\,[-(g=1)]\,(f<g)$ implies $f.<P.g$ when P stands for $f+(-g)$.

In other words, given that f is not null, g not universally necessary, and that f implies g, then there is a setting P, namely $f+(-g)$, such that f dynamically implies g relative to P.

4.75. $fg\,\gamma\,h$ implies $f\gamma h$.

4.76. $(f\gamma g)\,(g<h)$ implies $f\gamma h$.

5. **Variable settings.** We now consider the properties of $f((g))$ and for the convenience of the reader we repeat the primitive assertions concerned.

2.31. $f<f((g))$.

2.32. $(f<g)\,(g \text{ rel } h)$ implies $f((h))<g$.

2.33. $f((g))$ rel g.

2.34. g rel h implies g rel $f((h))$.

5.11. $(f \text{ rel } h)\,(g \text{ rel } h)$ is equivalent to fg rel h, and to $(f+g)$ rel h.

Use 2.26, 2.23, 2.25 and 3.72, 3.73.

5.12. f rel g implies f rel gh.

5.13. g rel $f((g))$, from 2.34 and g rel g.

5.14. $(f<g)\,(g \text{ rel } h)$ implies $f((h)).<r.g$.

5.15. $g((h))$ rel $f((h))$.

5.16. $f<g((h))$ implies $f((h)).<r.g((h))$, from 2.32.

5.17. $f<g$ implies $f((h)).<r.g((h))$.

This is remarkable as proving that when there is the "ordinary" implication between f and g, there is "relevant implication" between $f((h))$ and $g((h))$ whatever h may be.

5.18. f rel g implies $f((g)) . < r . f$, from 5.14 and $f < f$.

5.19. f rel g implies $f((g)) = f$.

5.20. $f((f)) . = r . f$, from 5.18, f rel f, and 2.31, 2.33.

5.21. g rel h implies $f((g))$ rel h.

5.22. g rel h implies $f((h)) . < r . f((g))$,
using 5.21, and 5.14 with $f((g))$ in place of g.

5.23. $f = g$ implies $f((h)) . = r . g((h))$.

5.24. $(f$ rel $g)\,(g$ rel $f)$ implies $h((f)) \quad . = r . \quad h((g))$.

Hence when $f . = r . g$, we can replace f by g in either $f((h))$ or $h((f))$.

5.25. $f((gh)) . = r . f((g + h))$.

5.26. $f((g)) . = r . f((-g))$.

5.27. $f((gh)) . < r . f((g))$.

5.28. $f((gh)) . < r . f((g))\, f((h))$.

5.29. $f < g$ is equivalent to $f((g)) . < r . g$.

Use 5.17 and 2.31.

5.30. $h((g)) . = r . h\,[[f((g))]]$.

5.31. $[f((g))]\,((g)) . = r . [f((g))]\,[[f((g))]]$,
$. = r . f((g))$.

5.32. We introduce the definition,
$f << g$ denotes $[f < g((f))]\,(f$ rel $g)$,
and say that then f is a branch of g, when all the terms of f are contained in g and when f includes among its factor propositions all the propositions implied by g which are relevant in f. This may be better understood if f, g are replaced by the settings P, Q.

As examples, organic chemistry is a branch of chemistry, electricity is a branch of physics. On the contrary in a postulate system for projective geometry, the existence of three dimensions carries with it certain properties of the plane which are not determined by the postulates referring only to points in a plane. In this case the system consisting only of the latter is not a branch of the complete solid geometry. A branch may contain propositions not contained in the "whole" provided they are relevant in the whole.

While $fg . < r . f$ it is not true, in general, that $fg << f$.

5.33. $(f << g)\,(g << h)$ implies $f << h$.

If f is a branch of g and g of h, then f is a branch of h. For
$(f$ rel $g)\,(g$ rel $h)$ implies f rel h.
$[g < h((g))]\,(f$ rel $g)$ implies $g < h((f))$,
implies $g((f)) < h((f))$.

5.34. $(fg)\,((f)) . = r . f . g((f))$.

For

$$(fg)((f)) . < r . f((f)) . = r . f,$$
$$(fg)((f)) . < r . g((f)),$$
$$fg < (fg)((f)),$$
$$g < (-f) + (fg)((f)),$$

and this is relevant in f, so that

$$g((f)) < (-f) + (fg)((f)),$$
$$fg((f)) < (fg)((f)).$$

Also the right-hand expression is relevant with respect to the left.

The ordinary product, fg, contains any terms which are in either f or g. There is some value in a product which contains only terms contained in one of the factors. This product is no longer symmetrical.

5.35. $fg((f)) << f$.

For f and $g((f))$ are relevant in f and

$$fg((f)) < f = f((f)) = f[[(fg)((f))]].$$

5.36. $fg((f)) << g((f))$.

For $fg((f))$ rel f rel $g((f))$ and

$$[g((f))][[(fg)((f))]] = [g((f))]((f)) = g((f)).$$

There appears to be no theorem stating that, when f is a branch of g and of h, it is a branch of fg, or that it is a branch of $g\,h((g))$.

5.37. $(fg)((h)) < f((h))\,g((h))$.

For

$$fg < f \text{ implies } (fg)((h)) < f((h)),$$

and similarly with $g((h))$.

5.38. $f((h)) = f$ is equivalent to $f((h))(-f)((h)) = 0$.

$$f((h)) < f \text{ implies } -f < -f((h)),$$
$$\text{implies } (-f)((h)) < -f((h)),$$
$$\text{implies } f((h))(-f)((h)) = 0.$$
$$f((h))(-f)((h)) = 0 \text{ implies}$$
$$f((h))(-f) = 0, \text{ implies } f((h)) < f.$$

5.39. $f((h)) = f$ is equivalent to $(-f)((h)) = -f$.

5.40. $f = 0$ implies $f((h)) = f$.

By 3.67, when $f = 0$,

$$f < -f((h)),$$
$$f((h)) < -f((h)),$$

since the right hand expression is relevant in h.

5.41. $f = 1$ implies $f((h)) = f$.

5.42. $(f = g)(g \text{ rel } h)$ implies $f((h)) = f$.

$$f = g \text{ implies } f((h)) = g((h)),$$
$$g \text{ rel } h \text{ implies } g((h)) = g.$$

Conversely, when $f((h)) = f$, there is a g, namely $f((h))$, such that $(f = g)$ $(g \text{ rel } h)$.

The relation $f((h)) = f$ has many of the properties of f rel h. Indeed it can be shown that all the assertions 2.21 to 2.34 are satisfied with $f((h)) = f$ in place of f rel h except 2.25. On the other hand,

$[f((g)) = f]\,(g \text{ rel } h)$ implies $f((h)) = f$.

5.43. $f((h)) = f$ implies $f((h))\,g((h)) = (fg)((h))$.

$fg < (fg)((h))$,

$g < (-f) + (fg)((h))$,

$f((h)) = f$ implies $g < -f((h)) + (fg)((h))$,

implies $g((h)) < -f((h)) + (fg)((h))$,

implies $f((h))\,g((h)) < (fg)((h))$,

and we use 5.37.

When

$$f((h))\,g((h)) = (fg)((h))$$

we may say that f, g are separably potent in h. We see that $f((h)) = f$ is equivalent to "f, $-f$ are separably potent in h". Also this relation holds between f, g and h when either f or g is null or necessary or when either is relevant in (or equivalent to a fragment itself relevant in) h.

On the other hand, there are cases where the relation does not hold. For example,

h denotes "astronomy",

f denotes "the sun is round and pleasant",

g denotes "the sun is hot and unpleasant".

$(fg)((h))$ is null, since fg is null.

$f((h))\,g((h))$ is "the sun is round and hot" which we have every reason to believe is not null. Again

h is an Abelian group postulate system.

f is $(aob = boa)\,(aob > a)$,

g is "for some z, $zoz = z$ and not $(z > z)$".

We suppose that "$>$" is not relevant in h.

5.44. $[f((h))\,g((h)) = (fg)((h))]\,(h << f)\,(h << g)$ implies $(h << fg)$.

We could proceed to consider a further concept, denoted by $f((g, h))$ which would be the product of the fragments implied by f which are simultaneously relevant in g and h, and define a very narrow product,

$f \times g$ denoting $(fg)((f, g))$,

but, on the one hand, such a product might not exist and, secondly, the paper would become too long.

6. **Postulate systems.** It is our view that the ordinary propositional fragments of the more concrete type and the sets of such fragments which make up a branch of science obtain their apparently unique significance from

being considered in combination with a much greater set of propositional fragments left unexpressed but understood by the propounder and his audience. Moreover, in an experimental or observational science this setting continually grows by the addition of (logical multiplication) more and more fragments, which are sometimes relevant in the old setting, sometimes add really new terms.

A postulate system differs from such a concrete science in two ways. In the first place all tacit understandings (except the rules of logic) are debarred. Only the actually stated propositional fragments are to be considered. Secondly, and chiefly in order to satisfy the first requirement, no reference is allowed to any of the more special elements of experience such as colours, sounds, motions and so forth which are so interwoven with tacitly understood fragments that they could not be thoroughly divorced from them.

It follows from these distinctions that the variable character of a postulate is more explicit than that of a scientific propositional fragment.

There is another distinction which is, here, of less importance. This is that in science discussion is chiefly concerned with what is true (implied by the setting or by future settings as they arise through experiment or observation). In a postulate system it is only the logical relation between the parts of the system and of related systems which is of interest.

6.1. The first condition which must be satisfied by a postulate system is that it be self-consistent. Remembering that

$P = 0$ denotes $P < -P$, this condition is equivalent to

$P \gamma P$.

6.2. A second condition which is not necessary but of great importance when it is satisfied is that of closure.

We say that a postulate system, P, is closed when every propositional fragment which is compatible with and relevant in P is implied by P, that is,

(P closed) denotes $[(f \gamma P)\ (f \text{ rel } P) \text{ implies } P < f]$.

6.3. $(f \text{ rel } P)\ (P \text{ closed})$ implies not-$(f . < P . g)$.

When P is closed,

$(f \text{ rel } P)\ (f . < P . g)$ implies $(f \text{ rel } P)\ (f \gamma P)\ (fP < g)$,
implies $(P < f)\ (fP < g)$,
implies $P < g$.

$f . < P . g$ implies $(-g) \gamma P$,
implies not-$(P < g)$,

which leads to a self-contradiction. Hence, when P is closed,

$f \text{ rel } P$ implies not-$(f . < P . g)$.

When a postulate system is closed it cannot be effectively extended without introducing a new symbol or operation distinct from those already used.

6.4. A third condition for a postulate system is that its parts should be independent, that is, that no postulate should be implied by the others together.

The study of this property requires us, however, to regard a postulate system not as a single product but as a complex for which we have not set up the necessary machinery. Furthermore this question, like that of simplicity, is dependent on artistic form and does not affect the "object" determined by the system as a whole.

6.5. Huntington* has defined the sufficiency of a postulate-set P for a system (K, o), in which K refers to a class and o to a rule of combination.

The set P is said to be sufficient when all systems (K, o) which satisfy it are necessarily isomorphic. (K, o), (K', o') are said to be isomorphic when the elements of K and K' can be put into a one-to-one correspondence such that each a in K corresponds to a' in K', b to b' and aob to $a'o'b'$.

In place of the class K let us use its defining propositional function Kx. We also consider, instead of the rule of combination o, a three-term relation $Oyzu$ which stands for $yoz = u$. (K, o) (K', o') are isomorphic when (and only when) a correspondence xRx', yRy', $\cdots$ exists such that

Kx is equivalent to $K'x'$, $Oyzu$ equivalent to $O'y'z'u'$.

If they are isomorphic then any fragment relevant in P, that is, a logical function of K, o only, say $\psi(K, o)$ must be such that

$\psi(K, o)$ is equivalent to $\psi(K', o')$.

[Note: In ψ the variables $xyzu$ are apparent only].

If they are not isomorphic, whatever correspondence xRx' is considered, either Kx is not equivalent to $K'x'$ or $Oyzu$ is not equivalent to $O'y'z'u'$ and there is a relevant fragment which does not have the same truth-value for (K, o) and (K', o').

If the set P is sufficient, when P is satisfied, all the systems are isomorphic and any relevant fragment

f denoting $f(K, o)$

has the same truth-value for all values of the variables, so that it is either null or necessary relative to P.

(P sufficient) implies

[f rel P implies $P < f$ or $P|f$]

implies [(f rel P) ($f\gamma P$) implies ($P < f$)], which is (P closed).

If the set P is not sufficient all the systems are not isomorphic, there is some f which is relevant and neither null nor necessary,

not-(P sufficient) implies

(for some f) [f rel P, $f\gamma P$, $-f\gamma P$],

implies not-[(all f) (f rel P, $f\gamma P$ implies not-($-f\gamma P$)].

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* Monographs on Topics of Modern Mathematics, IV, p. 171.

But

not-$(-f\gamma P)$ is equivalent to $P<f$.

Therefore

not-(P sufficient) implies not-(P closed).

Thus, if this not very clear reasoning is valid, closure and sufficiency are equivalent concepts. In the place cited Huntington gives an example of a sufficient set. We add here an example of a closed (and consistent and independent) set of postulates which possess no class as a variable.

Consider the three variables α, β, $<$, and the postulates,

(1) $\alpha < \beta$, (2) $\alpha < \alpha$,

(3) not-$(\beta < \alpha)$, (4) $\beta < \beta$.

The system is closed since the variable $<$ must couple two variables, α or β, by tacit understanding [A postulate such as $\alpha <$ or $\alpha\beta <$ or $\alpha <(<)$ would be regarded as irrelevant] and all four combinations have been asserted or negatived.

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THETA FUNCTIONS AND ARITHMETIC.

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The theta functions of p arguments and their allied multiply periodic quotients offer an inexhaustible store of relations between the numbers representing given sets of integers simultaneously in several systems of higher forms of any degree in any numbers of variables. Conversely the theory of such arithmetical relations implies that of the relations between theta functions, so that when either the theta or the arithmetical relations are given the others can be readily inferred. Logically it is possible to work in either direction; the theory of the theta functions being highly developed it is more profitable to proceed thence to the arithmetic which at present is practically non-existent. In this paper we develop certain general methods for making the transition.

The case $p = 1$ (elliptic) was treated in some detail on a former occasion; a generalization of the method in this case will be published later.*

As an introduction to the case $p > 1$, we shall first still further extend the method in the case of $p = 1$. When $p > 1$ there is a radical change, described at the end of § 1, in the nature of the arithmetical results. The case $p = 2$ was sketched in a former paper.†

For simplicity in printing we consider when $p > 1$ only thetas with half integer characteristics. The treatment for theta functions of higher order proceeds along precisely similar lines.

1. The case $p = 1$.

1. It will be necessary first to restate in the form adapted to our present purpose two very special cases of a general result in the first paper cited.

Let f, g, h denote single valued functions of n independent variables x_j and let f be an even, g an odd function of all its variables simultaneously:

$$(1) \qquad f(-x_1, -x_2, \cdots, -x_n) = f(x_1, x_2, \cdots, x_n),$$

$$(2) \qquad g(-x_1, -x_2, \cdots, -x_n) = -g(x_1, x_2, \cdots, x_n).$$

* In the Transactions of the American Mathematical Society, as a sequel to the papers, ibid., 1921, vol. 22, p. 1; p. 198. Detailed applications to higher forms and generalized class number relations are contained in the paper (to be published later) read before the Society, Dec. 27, 1922.

† Amer. Journ. Math., vol. 44 (1922), pp. 1–11.

Beyond these restrictions f, g, h are arbitrary. Note in particular that h is not necessarily even or odd.

Let the k_i denote numerical constants, the c_{ij} integers (zero is an integer), and suppose that we have established the relations

$$\sum_i k_i f(c_{i1}, c_{i2}, \cdots, c_{in}) = 0, \tag{3}$$

$$\sum_i k_i g(c_{i1}, c_{i2}, \cdots, c_{in}) = 0. \tag{4}$$

Then (3), (4) together imply

$$\sum_i k_i h(c_{i1}, c_{i2}, \cdots, c_{in}) = 0. \tag{5}$$

For since f, g are arbitrary to the extent defined they may be replaced in (3), (4) by $\frac{1}{2}h(c_{i1}, c_{i2}, \cdots, c_{in}) \pm \frac{1}{2}h(-c_{i1}, -c_{i2}, \cdots, -c_{in})$ respectively, the upper sign giving a function satisfying (1), the lower one satisfying (2). Adding the results thus obtained from (3), (4) we get (5).

Again, it was shown that if the following is an identity in $y_1, y_2, \cdots, y_n$,

$$\sum_i k_i \cos(c_{i1}\, y_1 + c_{i2}\, y_2 + \cdots + c_{in}\, y_n) = 0, \tag{6}$$

then (6) implies (3); similarly the identity

$$\sum_i k_i \sin(c_{i1}\, y_1 + c_{i2}\, y_2 + \cdots + c_{in}\, y_n) = 0 \tag{7}$$

implies (4). Hence (6), (7) together imply (5). Conversely it is obvious that (3) implies (6) and that (4) implies (7). Hence (3), (6) are formally equivalent, likewise (4), (7), and (5) is formally equivalent to (6) and (7).

When $p > 1$ the theta functions of p arguments lead to arithmetical relations of the type (5). When $p = 1$ the relations are of the types (3), (4) or of a considerable extension of these types in which, however, a generalized eveness or oddness (parity) of the functions plays an essential part. The parity aspects for $p = 1$ having been fully discussed in the papers cited we need not further consider them here. The distinction between the case $p = 1$, $p > 1$ is that in the latter parity is irrelevant.

2. The further extension of the case $p = 1$ mentioned in the introduction is a consequence of relations of type (3). There appears to be no similar extension of type (4).

Denote by $R(z_1, z_2, \cdots, z_r)$ a rational integral algebraic function of $z_1, z_2, \cdots, z_r$ with integral coefficients. We shall call R a form or order r.

Note that R is not necessarily homogeneous. If integers z_i' $(i = 1, 2, \cdots, r)$ exist such that, for the fixed integer N,

$$R(z_1', z_2', \cdots, z_r') = N,$$

N is said to be represented in R by $(z_1', z_2', \cdots, z_r')$, and $R(z_1', z_2', \cdots, z_r')$ is called a representation of N in R.

A form of order r is called even or odd according as it does not or does change sign when the signs of all its variables are changed simultaneously. Thus R is even or odd according as

$$R(-z_1, -z_2, \cdots, -z_r) = \pm R(z_1, z_2, \cdots, z_r).$$

In general a form of order R is neither even nor odd.

Let R be an odd form, and suppose that $R(z_1', z_2', \cdots, z_r')$ is a representation of N in R. Then $R(-z_1', -z_2', \cdots, -z_r')$ is a representation of $-N$ in R. Suppose that M is not represented in R; that is, for no z_i'' is $R(z_1'', z_2'', \cdots, z_r'') = M$. Then $-M$ is not represented in R. For if it were, by what immediately precedes, we should have M represented in R. Hence either both or neither of $\pm N$ are represented in the odd form R.

Let $\varphi_R(N)$, where R is any form of order r, denote 1 or 0 according as N is or is not represented in R.

From the above it follows that if R is an odd form, $\varphi_R(N)$ is an even function of N.

Let $R(z)$ be an even form of order 1, and A, B constants, arbitrary except for the restriction $A < B$. Then if $R(z')$ lies between A and B, so also does $R(-z')$; while if $R(z')$ does not lie between A and B, neither does $R(-z')$. Hence if $\psi_{ARB}(N)$ denotes 1 or 0 according as $R(N)$ does or does not lie between A and B, $\psi_{ARB}(N)$ is an even function of N.

3. Next let O_i denote an odd form of order ω_i, and write

$$\prod_{i=1}^{n} \varphi_{O_i}(N_i) \equiv \Phi(N_1, N_2, \cdots, N_n),$$

so that Φ is an even function of each of its n variables and therefore also of all n simultaneously. Similarly for Ψ defined by

$$\prod_{i=1}^{n} \psi_{A_i E_i B_i}(N_i) \equiv \Psi(N_1, N_2, \cdots, N_n),$$

where E_i is an even form of order 1.

Write

$$f'(z_1, z_2, \cdots, z_n) \equiv \Phi(z_1, z_2, \cdots, z_n) f(z_1, z_2, \cdots, z_n),$$

where f is as in § 1. Then f' satisfies (1), and therefore (6) implies (3) with f' in place of f:

$$\sum f'(c_{i1}, c_{i2}, \cdots, c_{in}) = 0, \tag{8}$$

and therefore also in the same way, on writing

$$f''(z_1, z_2, \cdots, z_n) \equiv \Psi(z_1, z_2, \cdots, z_n) f'(z_1, z_2, \cdots, z_n),$$

$$\sum f''(c_{i1}, c_{i2}, \cdots, c_{in}) = 0. \tag{9}$$

4. It was shown in the papers cited, and illustrated by numerous examples,* that the c_{ij} in relations of type (3) arising from the single theta and elliptic functions are linear functions of the integers representing a set of arbitrary constant integers in a system S of quadratic forms. Comparing (3) with (8) we see that to S are now adjoined n forms of orders ω_i $(i = 1, \cdots, n)$, and that the c_{ij} in (3) are restricted in (7) to such values as are represented respectively in the n adjoined forms. Similarly (9) adds the condition that the c_{ij} in (8) are now restricted in magnitude in the manner indicated by Ψ. Obviously (9) includes (8) as a special case, which in turn includes (3). To pass from (9) to (8) we put $A_i = 0$, $B_i = \infty$ $(i = 1, \cdots, n)$ in Ψ; to pass from (8) to (3) we choose $\omega_i = 1$, $O_i(z_1) = z_1$ $(i = 1, \cdots, n)$.

When each odd form O_i in Φ is of the type cw^μ, where c, μ are integers, $c \not\equiv 0$, $\mu > 0$ and odd, it is possible to omit entirely all reference to the system S of quadratic forms, and we have relations of the kind (9) in direct reference to systems of higher forms. This case has been treated in detail and applied to class number relations in two of the papers mentioned.

Finally we note that $f(z_1, z_2, \cdots, z_n)$ may be replaced if desired by $k(z_1, z_2, \cdots, z_n)$, a single valued function even separately in each of its variables $z_i (i = 1, \cdots, n)$, since this function obviously satisfies (1).

II. The case $p > 1$.

5. Each of g_{jr}, h_{jr} $(j = 1, \cdots, p;\ r = 1, 2, 3, \cdots, 2^{2p})$ denotes a definite one of 0, 1; each $n_{jr} \gtreqless 0$ is an integer, $\nu_{jr} = 2n_{jr} + g_{jr}$, and L is an

* For illustrations involving pairs of quadratic forms see Giornale di Mat., vol. 59 (1921), pp. 1–22.

arbitrary single valued function of $\frac{1}{2}p(p+1)$ arguments. With the ν_{jr} $(j = 1, \cdots, p)$ and their binary products $\nu_{jr}\,\nu_{kr}$ $(j, k = 1, \cdots, p, j \neq k)$ as arguments we shall construct 2^{2p} functions $l(\nu_r)$ and $2^{p-1}(2^p+1)$ functions $\lambda(\nu_r)$, each having $\frac{1}{2}p(p+1)$ arguments. To simplify the printing we introduce the symbol $[h_r\,\nu_r]$,

$$[h_r\,\nu_r] \equiv [h_{1r}\,\nu_{1r},\ h_{2r}\,\nu_{2r},\ \cdots,\ h_{pr}\,\nu_{pr}],$$

defined by the properties

$$[h_r\,\nu_r]\times[h_s\,\nu_s] = [h_{1r}\,\nu_{1r}+h_{1s}\,\nu_{1s},\ h_{2r}\,\nu_{2r}+h_{2s}\,\nu_{2s},\ \cdots,\ h_{pr}\,\nu_{pr}+h_{ps}\,\nu_{ps}],$$

$[h_r\,\nu_r] = 1, i, -1, -i (i = \sqrt{-1})$, according as $\sum_{j=1}^{p} h_{jr}\,\nu_{jr} \equiv 0, 1, 2, 3 \bmod 4$.

The *characteristic*

$$\begin{bmatrix} g_r \\ h_r \end{bmatrix} \equiv \begin{bmatrix} g_{1r} & g_{2r} & \cdots & g_{pr} \\ h_{1r} & h_{2r} & \cdots & h_{pr} \end{bmatrix}$$

is called even or odd according as $\sum_{j=1}^{p} g_{jr}\,h_{jr}$ is even or odd. The function

$$[h_r\,\nu_r]\,L(\nu_{1r},\ \nu_{2r},\ \cdots,\ \nu_{pr},\ \nu_{1r}\,\nu_{2r},\ \cdots,\ \nu_{1r}\,\nu_{pr},\ \nu_{2r}\,\nu_{3r},\ \cdots,\ \nu_{p-1,r}\,\nu_{p,r})$$

is denoted by $l\begin{bmatrix} g_r \\ h_r \end{bmatrix}(\nu_r)$, or simply $l(\nu_r)$, and this function is called quasi even or quasi odd according as its characteristic $\begin{bmatrix} g_r \\ h_r \end{bmatrix}$ is even or odd. The function obtained from quasi even $l(\nu_r)$ on replacing each of the first p arguments in L by zero is denoted by $\lambda\begin{bmatrix} g_r \\ h_r \end{bmatrix}(\nu_r)$ or $\lambda(\nu_r)$, and is called a quasi constant. Transposing the theory of theta characteristics* to the present discussion we see that there are $2^{p-1}(2^p+1)$ quasi even $l(\nu_r)$, and hence the same number of quasi constants $\lambda(\nu_r)$, and $2^{p-1}(2^p-1)$ quasi odd $l(\nu_r)$.

6. Let t, τ be constant integers, $t \geqq 1$, $\tau \geqq 0$, and n_r $(r = 1, \cdots, p)$ arbitrary constant integers >0. The $\nu_{rj} = 2n_{rj} + g_{rj}$ are as already defined in § 5. Consider all the solutions ν_{rj} of the system of p independent equations

$$n_r = \sum_{j=1}^{t+\tau} \nu_{rj}^2 \quad (r = 1, 2, \cdots, p), \tag{10}$$

* Krazer, Lehrbuch der Thetafunktionen, Zweiter Teil (Leipzig 1903); or Baker, Abelian Functions (Cambridge, 1897).

and put

$$[h] \equiv \prod_{j=1}^{t+\tau} [h_j\ \nu_j], \qquad \alpha_r \equiv \sum_{j=1}^{t} \nu_{rj} \qquad (r = 1, 2, \cdots, p),$$

$$\beta_{r,r+s} \equiv \sum_{j=1}^{t+\tau} \nu_{rj}\ \nu_{r+s,j} \qquad (r = 1, \cdots, p-1, \quad s = 1, 2, \cdots, p-r).$$

From these construct the function

$$[h]\, L(\alpha_1, \alpha_2, \cdots, \alpha_p, \beta_{12}, \beta_{13}, \cdots, \beta_{1p}, \beta_{23}, \cdots, \beta_{2p}, \cdots, \beta_{p-1,p}),$$

give in this to the α_r, $\beta_{r,r+s}$ all their possible values as determined by (10), (there will obviously be only a finite number of values for the α, β), take the sum of all such functions thus obtained and denote it by

$$(n_r) \mid l(\nu_1)\, l(\nu_2) \cdots l(\nu_t)\, \lambda(\nu_{t+1})\, \lambda(\nu_{t+2}) \cdots \lambda(\nu_{t+\tau}).$$

This is called the symbolic product, or simply product, of

$$l(\nu_j),\ \lambda(\nu_k) \qquad (j = 1, \cdots, t,\ k = t+1, \cdots, t+\tau)$$

with respect to the parameters $(n_r) \equiv (n_1, n_2, \cdots, n_p)$. If $\tau = 0$ the λ factors are absent. Obviously from the definitions this product is invariant under all permutations of its factors. Hence this kind of multiplication is commutative. It will be evident from § 8 in what senses it is also associative and distributive. When in the product the parameters (n_r) are understood we write

$$l(\nu_1) \cdots l(\nu_t)\, \lambda(\nu_{t+1}) \cdots \lambda(\nu_{t+\tau}) \equiv \mathcal{A}(\nu).$$

It is evident that $\mathcal{A}(\nu)$ is uniquely determined when (n_r) and the characteristics $\begin{bmatrix} g_r \\ h_r \end{bmatrix}$ $(r = 1, \cdots, t)$ and those, $\begin{bmatrix} g_s \\ h_s \end{bmatrix}$ $(s = t+1, \cdots, \tau)$, of the $l(\nu_r)$, $\lambda(\nu_s)$ respectively are assigned. Hence for (n_r) fixed there are possible

$$\frac{(2^{2p}!)^2}{t!\ \tau!\ (2^{2p}-t)!\ (2^{2p}-\tau)!}$$

products of t quasi even or quasi odd functions l and τ quasi constants λ. The degree of each such product is defined to be t. That is, in deter-

mining the degree of a product, the λ's are treated as absolute constants. A homogeneous linear relation between products of degree t is therefore defined. Let

$$\sum_{j=0}^{n} k_j \Delta_j(\nu) = 0 \tag{11}$$

be such a relation. We shall now show that this implies a more general relation.

7. In (11) the arguments $\alpha_r^{(j)}$, $\beta_{r,r+s}^{(j)}$ of L in $\Delta_j(\nu)$ are restricted only by equations of the type (10),

$$n_r = \sum_{s=1}^{t+\tau} \nu_{rs}^{(j)} \nu_s^{(j)} \qquad (r = 1, 2, \cdots, p). \tag{12}$$

In addition we now require $\alpha_r^{(j)}$ to be represented in the form R_r of order a_r, and $\beta_{r,r+s}^{(j)}$ to be represented in the form $R_{r,r+s}$ of order $b_{r,r+s}$:

$$\alpha_r^{(j)} = \sum_{s=1}^{t} \nu_{rs}^{(j)} = R_r, \quad \beta_{r,r+s}^{(j)} = \sum_{\sigma=1}^{t+\tau} \nu_{r\sigma}^{(j)} \nu_{r+s,\sigma}^{(j)} = R_{r,r+s} \quad (j = 1, \cdots, n), \tag{13}$$

and further from all the $\alpha_r^{(j)}$, $\beta_{r,r+s}^{(j)}$ satisfying (12), (13) we select as arguments of the function L occurring in $\Delta_j(\nu)$ only those such that

$$A_r < R_\rho'(\alpha_r^{(j)}) < B_r, \qquad A_{\rho,\rho+\sigma} < R_{\rho,\rho+\sigma}'(\beta_{r,r+s}^{(j)}) < B_{\rho,\rho+\sigma} \tag{14}$$

where the R' are forms of order 1 and the A, B ($A < B$ in each case) arbitrary constants. Denote the result of substituting in $\sum_{j=0}^{n} k_j \Delta_j(\nu)$ for the $\alpha_r^{(j)}$, $\beta_{r,r+s}^{(j)}$ as conditioned only by (12) the corresponding values conditioned by (12), (13), (14), by $\sum_{j=0}^{n} k_j \Delta_j'(n)$. Then by an argument similar to that in § 3 we see that (11) implies

$$\sum_{j=0}^{n} k_j \Delta_j'(\nu) = 0, \tag{15}$$

and that by the suitable simple assumptions concerning the R, A, B in (13), (14) we can recover (11) as a special case of (15). The degree of (15) is defined to be t(= the degree of each product), and (15) is called an extended product relation of degree t.

8. The characteristics of the l, λ in the product $A_j'(v)$ may be denoted for the moment by

$$\begin{bmatrix} g_r^{(j)} \\ h_r^{(j)} \end{bmatrix} \qquad (r = 1, 2, \cdots, t).$$

Consider the theta functions of p arguments v_r having the same characteristics, and in the last τ of these put $v_r = 0\,(r = 1, \cdots, p)$. Denote the product of the t theta functions and τ theta constants thus obtained by $\Theta_j(v)$, and suppose that

$$\sum_{j=0}^{n} k_j\,\Theta_j(v) = 0 \tag{16}$$

is a true relation. We shall next show that (16) implies (11) and hence (15). It will follow immediately by reversing the steps that (15) implies (16). Hence since any algebraic relation between thetas can be reduced to the form (16), it will follow that any such relation is formally equivalent to a relation between integers of the type (15).

9. In the general theta function, with $\tau_{rs} = \tau_{sr}$,

$$\vartheta\begin{bmatrix} g \\ h \end{bmatrix}^{(v)} \equiv \vartheta\begin{bmatrix} g_1 & g_2 & \cdots & g_p \\ h_1 & h_2 & \cdots & h_p \end{bmatrix}^{(v_1, v_2, \cdots, v_p)}$$

$$= \sum_{n_1, \cdots, n_p}^{-\infty, \infty} \exp \pi i \left[\sum_{r=1}^{p} \sum_{s=1}^{p} \tau_{rs} (n_r + \tfrac{1}{2} g_r)\,(n_s + \tfrac{1}{2} g_s) + 2 \sum_{r=1}^{p} (n_r + \tfrac{1}{2} g_r)\,(v_r + \tfrac{1}{2} h_r) \right],$$

write $2n_r + g_r = \nu_r$, $\pi v_r = u_r$, $(r = 1, 2, \cdots, p)$,

$$\tfrac{1}{2}\pi\tau_{r,r+s} = u_{r,r+s} \qquad (r = 1, 2, \cdots, p-1,\ s = 1, 2, \cdots, p-r),$$

and put $e^{i\pi\tau_{rr}/4} \equiv q_r (r = 1, \cdots, p)$. Then the outer $\sum$ referring to $n_r = -\infty$ to $\infty\,(r = 1, \cdots, p)$, we have

$$\vartheta\begin{bmatrix} g \\ h \end{bmatrix}^{(v)} = \sum \left\{ [h\nu]\, q_1^{\nu_1^2} q_2^{\nu_2^2} q_3^{\nu_3^2} \cdots q_p^{\nu_p^2} \exp i \left(\sum_{r=1}^{p} \nu_r u_r + \sum_{r=1}^{p-1} \sum_{s=1}^{p-r} \nu_r \nu_s u_{r,r+s} \right) \right\},$$

which is the standard form adapted to arithmetic. When $\begin{bmatrix} g \\ h \end{bmatrix}$ is even the theta constant obtained from this by putting $v_r = 0\,(r = 1, \cdots, p)$ is denoted by $\vartheta\begin{bmatrix} g \\ h \end{bmatrix}$.

Consider the product of t theta functions $\vartheta\begin{bmatrix} g_r \\ h_r \end{bmatrix}^{(v)} (r = 1, \cdots, t)$ and τ theta constants $\vartheta\begin{bmatrix} g_r \\ h_r \end{bmatrix} (r = 1, \cdots, \tau)$, the notation for the characteristics being as in § 5. Then in the notation of (10) it is evident that the coefficient of $q_1^{n_1} q_2^{n_2} \cdots q_s^{n_s}$ in the product is

$$\sum [h] \exp i\left(\sum_{r=1}^{p} \alpha_r u_r + \sum_{s=1}^{p-1} \sum_{s=1}^{p-r} \beta_{r,r+s} u_{r,r+s}\right),$$

the outer $\sum$ referring to all solutions of (10). Hence equating to zero the like coefficient in (16) we get

$$(17) \qquad \sum_{j=1}^{n} k_j \left\{\sum [h^{(j)}] \exp i\left(\sum_{r=1}^{p} \alpha_r^{(j)} u_r + \sum_{s=1}^{p-1} \sum_{s=1}^{p-r} \beta_{r,r+s}^{(j)} u_{r,r+s}\right)\right\} = 0.$$

Now a relation of the type

$$\sum_{j=1}^{n} k_j \exp i(\alpha_1^{(j)} u_1 + \alpha_2^{(j)} u_2 + \cdots \alpha_m^{(j)} u_m) = 0, \qquad (m = \tfrac{1}{2} p(p+1)),$$

implies by § 1 the pair of relations

$$\sum_{j=1}^{n} k_j f(\alpha_1^{(j)}, \cdots, \alpha_m^{(j)}) = 0; \qquad \sum_{j=1}^{n} k_j g(\alpha_1^{(j)}, \cdots, \alpha_m^{(j)}) = 0,$$

and therefore by § 1, L being as in § 5,

$$\sum_{j=1}^{n} k_j L(\alpha_1^{(j)}, \cdots, \alpha_m^{(j)}) = 0.$$

Hence (17), and therefore (16), implies (15).

A NEW TYPE OF CRITERIA FOR THE FIRST CASE OF FERMAT'S LAST THEOREM.*

By H. S. Vandiver.

In this paper it will be shown that if

$$u^p + v^p + w^p = 0 \tag{1}$$

is satisfied in integers u, v and w, not zero and prime to the odd prime p, then

$$1 + \frac{1}{2^2} + \cdots + \frac{1}{\left[\frac{p}{3}\right]^2} \equiv 0 \pmod{p}.$$

To obtain this result we transform the criteria of Kummer:

$$B_k f_{p-2k}(t) \equiv 0, \qquad f_{p-1}(t) \equiv 0, \pmod{p}, \tag{2}$$

where $k = 1, 2, \cdots, (p-3)/2$, $B_1 = 1/6$, $B_2 = 1/30, \cdots$ are the numbers of Bernoulli, and

$$f_r(t) = t + 2^{r-1} t^2 + \cdots + (p-1)^{r-1} t^{p-1}$$

for $r > 1$, and

$$f_1(t) = 1 + t + t^2 + \cdots + t^{p-1},$$

$$-t = u/v,\ v/u,\ v/w,\ w/v,\ u/w,\ w/u, \pmod{p}.$$

It follows that

$$t,\ \frac{1}{t},\ 1-t,\ \frac{1}{1-t},\ \frac{t-1}{t},\ \frac{t}{t-1},$$

all satisfy (2). These give six incongruent values modulo p unless $t^2 - t + 1 \equiv 0$ and $t = -1, 1/2, 2$. We have also†

$$\frac{m^n - 1}{n} b_n \equiv \sum{}' \frac{f_n(\varrho)}{\varrho^p - 1}, \pmod{p}, \tag{2a}$$

$n = 1, 2, \cdots, p-1$; $b_1 = -1/2$, $b_{2r+1} = 0$, $r > 0$, $b_{2r} = (-1)^{r+1} B_r$,

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* Presented to the American Mathematical Society, Sept. 7, 1923.

The writer was enabled to carry out this investigation through a grant, relieving him of the duties of teaching, from the Heckscher Foundation for the Advancement of Research established by August Heckscher at Cornell University.

† Vandiver, Crelle 144, 1914, p. 315, relation (9). This is true for any m prime to p, as is easily seen.

where ϱ is an mth root of unity, the sign $\sum'$ indicates the summation over all the distinct values of ϱ other than unity, m being any integer prime to p.

We shall use the obvious identity

$$f_1(x)\,f_1(y) = \frac{(x^p-1)\,f_1(y)}{x/y-1} - \frac{x/y\,(y^p-1)\,f_1(x)}{x/y-1}, \tag{3}$$

where x and y are indeterminates. Setting $x = xe^{hv}$, $y = ye^{kv}$, h and k being indeterminates not zero, multiplying through by $(x/y)^p - 1$, differentiating a times, $(a \leqq p-2)$, with respect to v, and putting $v = 0$, we have

$$A = B - C + pG, \tag{4}$$

where

$$A = \left(\left(\frac{x}{y}\right)^p - 1\right) \sum_{n=1}^{a+1} \binom{a}{n-1} k^{a+1-n} h^{n-1} f_n(x)\, f_{a+2-n}(y),$$

$$B = (x^p-1) \sum_{n=1}^{a+1} \binom{a}{n-1} k^{a+1-n} (h-k)^{n-1} f_n\left(\frac{x}{y}\right) f_{a+2-n}(y),$$

$$C = (y^p-1) \sum_{n=1}^{a+1} \binom{a}{n-1} k^{a+1-n} (h-k)^{n-1} f_{a+2-n}(x)\, f_n'\left(\frac{x}{y}\right),$$

G is an algebraic expression in x, y, h and k, and $f_1'(u) = u^p + u^{p-1} + \cdots + u$, $f_r'(u) = f_r(u)$ for $r > 1$.

Now in (4) set $x = xe^v$, $y = ye^v$, differentiate $p-2-a$ times with respect to v, and we then have for $v = 0$

$$A_1 = B_1 - C_1 + pG_1,$$

where

$$B_1 = (x^p-1) \sum_{n=1}^{a+1} \binom{a}{n-1} k^{a+1-n} (h-k)^{n-1} f_n\left(\frac{x}{y}\right) f_{p-n}(y),$$

$$C_1 = (y^p-1) \sum_{n=1}^{a+1} \binom{a}{n-1} k^{a+1-n} (h-k)^{n-1} f_n'\left(\frac{x}{y}\right) f_{p-n}(x),$$

A_1 is a linear combination of the quantities $f_r(x)\,f_{p-r}(y)$, $r = 1, 2, \cdots, p-1$, with coefficients which are polynomials in h and k with integral coefficients, multiplied by $(x/y)^p - 1$, and G_1 is an algebraic function of x, y, h and k.

Let $y = \varrho$, where ϱ is an mth root of unity, $\neq 1$, and $x = t$; then

$$\sum' \frac{A_1(t, \varrho)}{\varrho^p - 1} = B_2 - C_2 + G_2 p,$$

where

$$B_2 = (t^p-1)\sum{}' \sum_{n=1}^{a+1} \binom{a}{n-1} k^{a+1-n}(h-k)^{n-1} f_n\left(\frac{t}{\varrho}\right)\frac{f_{p-n}(\varrho)}{\varrho^p-1},$$

$$C_2 = \sum_{n=1}^{a+1}\sum{}' \binom{a}{n-1} h^{a+1-n}(h-k)^{n-1} f_n'\left(\frac{t}{\varrho}\right) f_{p-n}(t).$$

This gives

$$(5)\quad (t^{pm}-1)\sum{}' \frac{A_1(t,\varrho)}{((t/\varrho)^p-1)(\varrho^p-1)}$$

$$= \sum{}' \frac{(t^{pm}-1)}{(t/\varrho)^p-1} B_2 - \sum{}' \frac{(t^{pm}-1)}{(t/\varrho)^p-1} C_2 + G_3 p,$$

$$= B_3 - C_3 + G_3 p,$$

where

$$B_3 = (t^p-1)\sum{}' \sum_{n=1}^{a+1} \binom{a}{n-1} k^{a+1-n}(h-k)^{n-1} F_n\left(\frac{t}{\varrho}\right)\frac{f_{p-n}(\varrho)}{\varrho^p-1},$$

$$C_3 = \sum{}' \sum_{n=1}^{a+1} \binom{a}{n-1} h^{a+1-n}(h-k)^{n-1} F_n'\left(\frac{t}{\varrho}\right) f_{p-n}(t),$$

$$F_n(z) = \sum_{r=1}^{pm-1} r^{n-1} z^r, \quad (n>1), \qquad F_1(z) = \sum_{r=0}^{mp-1} z^r,$$

$$F_1'(z) = \sum_{s=1}^{pm} z^s, \qquad F_n' = F_n, \quad (n>1).$$

In (5) we note that $A_1(t,\varrho)$ is divisible by $(t/\varrho)^p-1$, and that ϱ^p-1 is prime to (p) in the algebraic field defined by a primitive mth root of unity. Hence, if h and k are integers, we may change (5) to a congruence with the ideal (p) as modulus in the algebraic field above mentioned, and use the Kummer criteria (2). Since

$$\sum{}' \frac{A_1(t,\varrho)}{((t/\varrho)^p-1)(\varrho^p-1)}$$

is a linear combination of expressions of the type

$$\sum{}' \frac{f_n(\varrho)}{\varrho^p-1} f_{p-n}(t), \qquad (n=1,2,\cdots,p-1)$$

then (5) becomes, after using (2) and (2a),

$$(5a) \qquad k^a (t^{pm}-1) \sum' \frac{f_1(t) f_{p-1}(\varrho)}{\varrho^p - 1} \equiv B_3 - C_3 \pmod{p}.$$

From this general relation a number of criteria may be obtained. In particular, for $a = 1$, it gives,

$$k(t^{pm}-1)\sum' f_1(t)\frac{f_{p-1}(\varrho)}{\varrho^p-1} \equiv (t^p-1)k\sum' F_1\left(\frac{t}{\varrho}\right)\frac{f_{p-1}(\varrho)}{\varrho^p-1}$$
$$+(t^p-1)(h-k)\sum' F_2\left(\frac{t}{\varrho}\right)\frac{f_{p-2}(\varrho)}{\varrho^p-1}$$
$$-h\sum' F_1\left(\frac{t}{\varrho}\right) f_{p-1}(t) - (h-k)\sum' F_2\left(\frac{t}{\varrho}\right) f_{p-2}(t) \pmod{p}.$$

For $h = k \not\equiv 0 \pmod{p}$, this becomes

$$(6) \quad (t^{pm}-1)\sum' f_1(t)\frac{f_{p-1}(\varrho)}{\varrho^p-1} \equiv (t^p-1)\sum' F_1\left(\frac{t}{\varrho}\right)\frac{f_{p-1}(\varrho)}{\varrho^p-1} \pmod{p},$$

which is the relation of Mirimanoff,* leading to the criteria $2^{p-1} \equiv 3^{p-1} \equiv 1 \pmod{p^2}$, for the solution of (1) in integers prime to p. Now in (6) put $h \equiv 0 \pmod{p}$ and $h \not\equiv k \pmod{p}$. From (2) we have for $k = 1$, $f_{p-2}(t) \equiv 0 \pmod{p}$. Hence (6) reduces to

$$(7) \qquad \sum' F_2\left(\frac{t}{\varrho}\right)\frac{f_{p-2}(\varrho)}{\varrho^p-1} \equiv 0 \pmod{p}.$$

For $m = 2$, the expression on the left is identically divisible by p. For $m = 3$ it may be written

$$(7a) \qquad \sum_{k=0}^{2} \sum' \frac{\varrho^{-k} f_{p-2}(\varrho)}{\varrho^p - 1} \left(\sum_{r=0}^{p-1} (3r+k)\, t^{3r+k}\right) \equiv 0 \pmod{p}.$$

To reduce this we note that

$$x + x^4 + \cdots + x^{3(p-1)+1} = x(x^3-1)^{p-1} + pJ(x),$$

* Crelle, 139, 1911, p. 317, relation (17).

where $J(x)$ is an integral function of x with rational integral coefficients. Set xe^v for x, differentiate with respect to v, and set $v = 0$; we have

$$\sum_{r=0}^{p-1} (3r+1)\, x^{3r+1} \equiv -(x+2x^4)(x^3-1)^{p-2} \pmod{p}.$$

Similarly

$$\sum_{r=0}^{p-1} (3r+2)\, x^{3r+2} \equiv -(x^5+2x^2)(x^3-1)^{p-2} \pmod{p}.$$

We have also, if we write

$$A_k = \sum' \frac{\varrho^{-k} f_{p-2}(\varrho)}{\varrho^p - 1},$$

$$A_0 + A_1 + A_2 = 0,$$

and from (2a)

$$A_0 \equiv 0 \pmod{p},$$

so that

$$A_1 + A_2 \equiv 0 \pmod{p}. \tag{8}$$

Then (7a) may be written

$$(t^3-1)^{p-2}((t+2t^4)A_1 + (t^5+2t^2)A_2) \equiv 0 \pmod{p},$$

which congruence holds if, and only if,

$$(t^3-1)((2t^3+1)A_1 + (t^4+2t)A_2) \equiv 0 \pmod{p}, \tag{9}$$

since $t \not\equiv 0 \pmod{p}$.

This must hold also for $1/t$ in lieu of t, so that we have

$$\left(\frac{1}{t^3}-1\right)\left(\left(1+\frac{2}{t^3}\right)A_1 + \left(\frac{1}{t^4}+\frac{2}{t}\right)A_2\right) \equiv 0 \pmod{p},$$

or

$$(t^3-1)((t^4+2t^2)A_1 + (2t^3+1)A_2) \equiv 0 \pmod{p}.$$

Comparing this with (9) we have

$$(t^3-1)(t^2-1)(t-1)^2(A_1 - A_2) \equiv 0 \pmod{p},$$

and, since $t \not\equiv 1 \pmod{p}$, we have

$$(t^3-1)(t+1)(A_1 - A_2) \equiv 0 \pmod{p}.$$

This congruence is of degree 4; hence it is satisfied by the values of t which also satisfy* $t^2 - t + 1 \equiv 0 \pmod{p}$, or the values $-1, 2$, and $1/2$, unless $A_1 - A_2 \equiv 0 \pmod{p}$. The congruence

$$(t^3 - 1)(t + 1) \equiv 0 \pmod{p}$$

is inconsistent with $t^2 - t + 1 \equiv 0 \pmod{p}$, and it is evidently not satisfied by $t = 2$ for $p > 7$. Hence

$$A_1 - A_2 \equiv 0 \pmod{p}$$

and comparison with (8) gives

$$A_1 \equiv A_2 \equiv 0 \pmod{p}. \tag{10}$$

By the Bernoulli summation formula we have

$$\frac{(x+b)^{n+1} - b_{n+1}}{n+1} = 1 + 2^n + \cdots + (x-1)^n, \tag{11}$$

where in the expansion of $(x+b)^{n+1}$, we substitute $b^c = b_c$.

Also†

$$(mb + l)^{p-2} - b^{p-2} \equiv 2 \sum{}' \frac{\varrho^l f_{p-2}(\varrho)}{\varrho^p - 1}.$$

Since $b^{p-2} = b_{p-2} = 0$, this may be written

$$(b + (l/m))^{p-2} \equiv 2m \sum{}' \frac{\varrho^{lj} f_{p-2}(\varrho)}{\varrho^p - 1} \pmod{p}, \tag{12}$$

where (l/m) represents the least positive solution j of $mj \equiv l \pmod{p}$. Then (11) gives

$$(b + (l/m))^{p-2} \equiv 2(1 + 2^{p-3} + \cdots + ((l/m) - 1)^{p-3}) \pmod{p}. \tag{13}$$

Let $m = 3$; we have $3j_1 = 1 + pk_1$, so that $j_1 - 1 = [pk_1/3]$, where $[x]$ is the greatest integer in x and $k_1 = 1$ or 2. Similarly $3j_2 = 1 + pk_2$ and $j_2 - 1 = [pk_2/3]$, $k_2 = 1$ or 2. If $k_1 = 1$, then $k_2 = 2$ and conversely. Hence, using (10), (12) and (13), we obtain

* According to Pollaczek, Wiener Bericht, 126, II a, 1917, 45–59, a root of $t^2 - t + 1 \equiv 0 \pmod{p}$ cannot satisfy (2).

† Frobenius, Berl. Sitzber., 1914, p. 655, formula (2) for $n = p - 2$.

$$1+2^{p-3}+\cdots+\left[\frac{p}{3}\right]^{p-3}\equiv 0 \pmod{p},$$

or

$$1+\frac{1}{2^2}+\cdots+\frac{1}{\left[\frac{p}{3}\right]^2}\equiv 0 \pmod{p}.$$

Hence we may state the following result.

If

$$x^p+y^p+z^p=0$$

is satisfied in integers x, y and z, not zero and prime to the odd prime p, then

$$1+\frac{1}{2^2}+\cdots+\frac{1}{\left[\frac{p}{3}\right]^2}\equiv 0 \pmod{p}.$$

This criterion has been tested for all primes less than 32 and is not satisfied in any case. It is possible that analogous criteria may be obtained by setting $m=4$, 5, etc., in (7) but this I have not attempted.

CORNELL UNIVERSITY,
July, 1923.

ALGEBRAIC FUNCTIONS AND THEIR DIVISORS.

By Gilbert Ames Bliss.

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Introduction. The theory of algebraic functions and their integrals is one of the most impressive of the contributions of the mathematicans of the nineteenth century, and one of the most beautiful of all the chapters in the domain of modern mathematics. It had its origin in the effort to generalize the theory of elliptic integrals of the form

$$I = \int \eta(x, y)\, dx,$$

where $\eta(x, y)$ is a rational function of x and y, and y is defined as a function of x by an equation of the form

$$y^2 - A_0 x^4 - A_1 x^3 - A_2 x^2 - A_3 x - A_4 = 0.$$

Early in the nineteenth century Abel conceived the idea of studying integrals of this type for which the variable y is defined as a function of x by means of a more general equation $f(x, y) = 0$ in which the first member may be any polynomial whatsoever in x and y instead of the relatively simple

one in the equation just given for the elliptic case. His extension of the problem seemed at first almost too great a generalization, but his results, notably the famous theorem now called by his name, indicated clearly that the theory of such integrals had a richness of content which justified elaborate and searching study. A function $y(x)$ defined by an equation $f(x, y) = 0$ such as has just been described is called an algebraic function, and the integrals I associated with it are called Abelian integrals.

There have been three principal methods of attacking the theory of algebraic functions and their integrals*. The first of these is based largely upon the researches of Abel (1826-9) and Riemann (1857), and is called the transcendental theory because in it the Abelian integrals play the central role. A second is the geometrical method, closely interwoven with the theory of higher plane curves. It was inaugurated by Clebsch and Gordan (1863-6), continued by Brill and Noether (1871), and presented more recently in attractive form by Severi (1914). The third and last method has been called, not any too appropriately, the arithmetic method. In contrast with the transcendental method its emphasis is placed primarily upon the construction and theory of rational functions $\eta(x, y)$ such as those which occur in the integrands of the integrals I, and only secondarily upon the integrals themselves. One of the earliest suggestions of such a theory is found in a paper which Kronecker presented to the Berlin Academy in 1862 but published first in 1881. More elaborate theories, differing widely in detail, are those of Dedekind and Weber (1882), Weierstrass (1902)†, and Fields (1906). The methods of Dedekind and Weber have been elaborated and improved by Hensel and Landsberg (1902) in their book above referred to.

The transcendental and the various arithmetic theories are in reality identical in purpose, though differing markedly in the mechanisms by means of which this purpose is carried out. One of the undesirable features of both the transcendental and geometrical theories has been the necessity of applying preliminary birational transformations in order to simplify the singular points of the curve represented by the fundamental algebraic equation $f(x, y) = 0$. In both cases the development of the theory is much

* An excellent description and comparison of these methods has been given in brief by Hensel and Landsberg, Theorie der algebraischen Funktionen, pp. 694–702. For more extensive summaries and references see Brill and Noether, Die Entwickelung der Theorie der algebraischen Funktionen, Deutsche Math. Ver., vol. 3, (1894), pp. 107–566; Wirtinger, Algebraische Funktionen und ihre Integrale, Encykl. d. math. Wiss., II B 2, pp. 115–175; Hensel, Arithmetische Theorie der algebraischen Funktionen, Encykl. d. math. Wiss., II C 5, pp. 533–674.

† The date is that of the publication of volume IV of Weierstrass' Werke. The theory there presented was developed in his lectures during the years 1875–6.

easier if only multiple points with distinct tangents, or better only double points with distinct tangents, are present. The arithmeticists have shown that one may succeed in attaining the same, and in many cases even more general, results without the assistance of such transformations. This has been done by Hensel and Landsberg in most interesting fashion in their treatise, and the purpose of the present paper is to give a relatively elementary introduction to their theory. The goals which it is planned to attain here are two, the construction of the three types of elementary integrals upon which the superstructure of the theory of Abelian integrals is based, and the proof of the Riemann-Roch Theorem which is fundamental for the theory of the rational functions $\eta(x, y)$.

It has always seemed to the writer to be unfortunate to have results of the importance of those which have just been mentioned buried so deeply in the text of the book of Hensel and Landsberg.* Before reaching them one must read through some three hundred pages dealing with these and many other matters. In the following sections only those theorems have been collected which form links in the chain of reasoning leading up to the principal results described in the last two sections of this paper. The theorems themselves will be familiar to those who have read Hensel and Landsberg's book, but the proofs are frequently fundamentally different and it is hoped simpler. This is most noticeably so perhaps in the establishment of Theorem 3, page 104, concerning the existence of a basis for every divisor, where the properties of the determinant Δ of page 103 turn out to be of great assistance, and in the discussion of the properties of complementary bases.

It is of course not possible to discuss completely any large portion of the theory of algebraic functions in a short paper of this sort. For this reason the properties of the fundamental expansions for an algebraic function have been described without proofs in Section 1, but the remainder of the paper is intended to be logically complete within itself. The expansions are familiar ones to those who have studied the theory of algebraic functions. It is hoped that they are here described with sufficient clearness, so that others may, after accepting them, gain some idea of the content of this very beautiful theory.

1. **Expansions for an algebraic function.** Let $f(x, y)$ be an irreducible polynomial in x and y with coefficients which are either real or complex. By an irreducible polynomial is meant one which is not decomposable into a product of factors of the same sort. If the degree of

* This feeling was emphasized by the necessity of referring to the Riemann-Roch Theorem in the footnote on page 277 of the author's paper, Birational transformations simplifying singularities of algebraic curves, Trans. Amer. Math. Soc., vol. 24 (1922), p. 274.

the function $f(x, y)$ in y is n, then the n values of y which satisfy the equation $f(x, y) = 0$ form what is called an n-valued algebraic function of x.

The theory of such algebraic functions is based upon the fact that near every value $x = x_0$ the roots of the equation $f = 0$ are expressible by a number of series of the form*

$$y = b(x-x_0)^{\frac{\mu}{r}} + b'(x-x_0)^{\frac{\mu'}{r}} + \cdots \tag{1}$$

where r is a positive integer and the increasing integers $\mu < \mu' < \cdots$ are either positive or negative. The numbers $r, \mu, \mu', \cdots$ are relatively prime so that the particular series (1) furnishes r distinct values of y when the r values of the root $(x-x_0)^{\frac{1}{r}}$ are substituted. The number of series corresponding to different values $x = x_0$ may be different, but the sum of the integers r for each $x = x_0$ is exactly n, corresponding to the fact that for each value of x near $x = x_0$ the equation $f = 0$ has n roots.

Near $x = \infty$ the expansions giving the values of y are in powers of $1/x$ instead of $x-x_0$. We may include both cases in the formula

$$y = bt^{\mu} + b't^{\mu'} + \cdots \tag{2}$$

where t is now either $(x-x_0)^{\frac{1}{r}}$ or $(1/x)^{\frac{1}{r}}$. Such an expansion is called a cycle and its order is defined to be the number $r-1$. There are only a finite number of cycles for which $r > 1$ and these are called branch cycles. By a transformation $x' = 1/(x-x_1)$ we can always bring it about that the cycles for $x = \infty$ have all values $r = 1$, and we shall in the following pages always suppose that this has been done unless expressly indicated otherwise.

The cycles (2) are the fundamental elements of the algebraic function. We shall not in this paper make use of the notion of the Riemann surface of the function, but if we were to do so it would be found that to each such cycle there corresponds a unique place on the Riemann surface, and conversely. For this reason we may represent such a cycle by the symbol P, and if desirable, refer to it also as a place.

Consider now an arbitrary rational function $\eta(x, y)$ of the variables x and y. At each cycle P it has an expansion

$$\eta(x, y) = ct^{\nu} + c't^{\nu'} + \cdots \tag{3}$$

* See for example Picard, Traité d'Analyse, vol. 2 (1893), chapt. XIII; Appel and Goursat, Théorie des fonctions algébriques, chapt. IV; Hensel and Landsberg, Theorie der algebraischen Functionen, pp. 39-52.

found by substituting in $\eta(x, y)$ the value $x = x_0 + t^r$ (or $x = 1/t^r$) and the series (2) for y. The numbers $\nu < \nu' < \cdots$ are positive or negative integers, and the smallest one ν is called the order of the function $\eta(x, y)$ at the cycle P. There are only a finite number of cycles at which ν is different from zero. These are called zeros of $\eta(x, y)$ when ν is positive and poles when ν is negative.

There are two expressions associated with every rational function $\eta(x, y)$ which are important for the developments of the following pages. These are the so-called *norm* and *trace* of η defined by the equations

$$N(\eta) = \eta(x, y_1) \cdots \eta(x, y_n),$$
$$T(\eta) = \eta(x, y_1) + \cdots + \eta(x, y_n),$$

where $y_1, \cdots, y_n$ are the n roots of the equation $f(x, y) = 0$ corresponding to the value x. The values $\eta(x, y_i)$ of η corresponding to the different values y_i are called *conjugate values* of η. The norm and the trace are both representable as rational functions of x since they are symmetric in the roots $y_1, \cdots, y_n$ and therefore expressible rationally in terms of the coefficients of the powers of y in the equation $f = 0$.

By means of the norm we can prove the well-known and useful theorem that *the sum of the orders of a rational function $\eta(x, y)$ is zero.* The r values of $\eta(x, y)$ corresponding to the r values of y defined by a cycle of the form (2) for a finite value $x = x_0$ are found by substituting in the expansion (3) the r values of the root $t = (x - x_0)^{\frac{1}{r}}$. Since the product of these r values of $\eta(x, y)$ occurs in the norm it follows that $N(\eta)$ has corresponding to each cycle P the factor $(x - x_0)^{\nu}$, and that its order at $x = x_0$ is the sum of the orders ν of $\eta(x, y)$ at the cycles for $x = x_0$. Similar remarks hold of course for the cycles for which $t = (1/x)^{\frac{1}{r}}$. We know that the sum of the orders of a rational function of x is zero, this being true in particular for $N(\eta)$, and the same result must therefore hold for the function $\eta(x, y)$.

The residue of a function $\eta(x, y)$ at a cycle (2) is defined to be the coefficient of $1/t$ in the expansion for the product $\eta\, dx/dt$, or, what is the same thing, it is the value of the integral

$$\frac{1}{2\pi i} \int_{\Gamma} \eta \frac{dx}{dt} dt$$

taken around a circle Γ in the t-plane with center at the origin $t = 0$. Since when t describes this circle once the value $x = x_0 + t^r$ describes r

times a circle C about $x = x_0$ in the x-plane it follows that the residue is also expressible as an integral

$$\frac{1}{2\pi i}\int_C [\eta(x, y_1) + \cdots + \eta(x, y_r)]\, dx$$

where $y_1, \cdots, y_r$ are the r values of y corresponding to the cycle (2). Evidently the residue at $x = x_0$

$$\frac{1}{2\pi i}\int_C T(\eta)\, dx$$

of the rational functions of x designated by $T(\eta)$ is the sum of the residues of $\eta(x, y)$ at the cycles corresponding to $x = x_0$. Since the sum of the residues of a rational function of x is zero we now have the result that *the sum of the residues of a rational function $\eta(x, y)$ is also zero.*

If a rational function $\eta(x, y)$ has no pole it must be a constant. For the symmetric functions of the conjugate values $\eta(x, y_i)$ are then all rational in x and everywhere finite, and hence by a well known theorem are equal to constants. Consequently these values $\eta(x, y_i)$ are the roots of a polynomial with constant coefficients. It is easy to show on the Riemann surface of the algebraic function that these constants have all the same value, since there is always a path on the surface along which $\eta(x, y)$ varies continuously from the value $\eta(x, y_i)$ to the value $\eta(x, y_k)$.

A final remark concerning the expansions for the trace $T(\eta)$ will be helpful. The sum of the r values defined by the series (3) and corresponding to a cycle (2) has an expansion beginning with the term

$$(1 + \omega^\nu + \omega^{2\nu} + \cdots + \omega^{(r-1)\nu})\, c\, t^\nu,$$

where ω is a primitive r-th root of unity. This is true because the r values of the root t are exactly the values $\omega^k t$ $(k = 0, 1, \cdots, r-1)$. The value of the parenthesis is r when ν is an integral multiple of r, since then each term in it is unity, but zero otherwise since the sum of the roots of unity of every index r is zero. *If the exponent ν/r for a particular cycle at $x = x_0$ is smaller than the similar exponents of all the other cycles corresponding to $x = x_0$, then the first term in the expansion of $T(\eta)$ in powers of $(x - x_0)$ will have exactly the exponent ν/r when this quotient is an integer, and a larger exponent when it is a fraction.*

2. **Bases for all rational functions.** A set of functions $\eta_k(x, y)$ $(k = 1, \cdots, n)$ is called a *basis for the totality of rational functions $\eta(x, y)$* if the determinant $|\eta_k(x, y_i)|$ of their conjugate values is not identically zero.

THEOREM 1. *Every rational function $\eta(x, y)$ is expressible in terms of a basis in the form*

$$\eta = R_1(x)\eta_1 + \cdots + R_n(x)\eta_n,$$

where the coefficients $R_k(x)$ are rational in x, and there is only one such expression for η.

To prove this we notice in the first place that a function $\zeta(x, y)$ for which the traces

$$T(\zeta\eta_k) = \zeta(x, y_1)\eta_k(x, y_1) + \cdots + \zeta(x, y_n)\eta_k(x, y_n) \qquad (k = 1, \cdots, n)$$

of the products $\zeta\eta_k$ all vanish identically must itself be identically zero, since the determinant $|\eta_k(x, y_i)|$ does not vanish identically. Since the determinant $|T(\eta_i\eta_k)| = |\eta_k(x, y_i)|^2$ is not identically zero the equations

$$T(\eta\eta_i) = R_1 T(\eta_i\eta_1) + R_2 T(\eta_i\eta_2) + \cdots + R_n T(\eta_i\eta_n) \qquad (i = 1, \cdots, n)$$

determine the coefficients $R_1, \cdots, R_n$ uniquely as rational function of x in such a way that the traces of the products $\zeta\eta_k$ for the function

$$\zeta = \eta - R_1\eta_1 - \cdots - R_n\eta_n$$

all vanish identically. Hence ζ is identically zero and η is expressible uniquely as described in the theorem.

It is clear that the determinant $|T(\eta_i\eta_k)| = |\eta_k(x, y_i)|^2$ for a basis is symmetric in $y_1, \cdots, y_n$ and therefore rational in x. Since it is not identically zero, it can have only a finite number of poles and zeros. At all other values of x it is different from zero.

If $\zeta_1, \cdots, \zeta_n$ are expressible in terms of a basis in the form

$$\zeta_k = R_{ki}\eta_1 + \cdots + R_{kn}\eta_n, \qquad (k = 1, \cdots, n),$$

then the determinant $|\zeta_k(x, y_i)|$ is the product of $|R_{ik}|$ and $|\eta_k(x_1 y_i)|$ and a necessary and sufficient condition that $\zeta_1, \cdots, \zeta_n$ form a basis is that the determinant $|R_{ik}|$ of the rational functions $R_{ik}(x)$ be different from zero.

A special case of a basis is the set of functions $1, y, \cdots, y^{n-1}$. The determinant of the conjugates of these powers of y is the product of the differences of the roots y_i $(i = 1, \cdots, n)$, as is well known, and it can not vanish identically since the roots of an irreducible polynomial $f(x, y)$ are

distinct except at special values of x. All other bases are obtainable from this one by linear transformations such as are described in the last paragraph.

3. Divisors and their bases. If $P_1, \cdots, P_s$ are distinct cycles for an algebraic function and $\mu_1, \cdots, \mu_s$ a corresponding set of positive or negative integers, then the expression $Q = P_1^{\mu_1} \cdots P_s^{\mu_s}$ is called a *divisor* and the sum $q = \mu_1 + \cdots + \mu_s$ is called the order of the divisor. A rational function $\eta(x, y)$ is a *multiple of the divisor* Q if its order at each cycle P_k is greater than or equal to μ_k, and if its orders at all other cycles of the algebraic function are greater than or equal to zero. The problem of determining the multiples of a divisor and their properties is a fundamental one for the theory of algebraic functions, as we shall see in the following pages. When it has been solved the determination of the Abelian integrals of various types associated with the algebraic function, and the proof of the important Riemann-Roch Theorem, are relatively simple matters.

As a preliminary to the determination of the multiples of a divisor Q we may study those rational functions $\eta(x, y)$ which have the properties of multiples except at the cycles corresponding to $x = \infty$ where no restriction whatever is now placed upon their behavior. A function of this sort is called a *multiple of* Q *except at infinity*, and the totality of such functions constitute the *ideal of* Q which may be denoted by the symbol $I(Q)$. In the determination of such an ideal only the places of Q which correspond to finite values $x = x_0$ have any effect, and we may without loss of generality suppose that Q contains only such places.

A *basis for the divisor* Q is a basis whose elements $\eta_1, \cdots, \eta_n$ are multiples of Q except at infinity, and which has the further property that the totality of such multiples is identical with the totality of functions $\eta(x, y)$ expressible in the form

$$\eta(x, y) = g_1(x)\eta_1 + \cdots + g_n(x)\eta_n, \tag{4}$$

where the coefficients $g_k(x)$ are polynomials.

It is not a priori evident that there will be a basis with these properties for every divisor Q, and one of our first tasks will be to prove that such a basis exists. Before attempting the proof, however, it will be useful to deduce a characteristic property of such a basis. To do this let us consider a finite value x_0 where the algebraic function $y(x)$ has three cycles A, B, C providing, respectively, a, b, c values for $y(x)$, and at which the orders required by Q are λ, μ, ν. The methods to be used would be quite analogous if there were more or fewer than three cycles for $x = x_0$, but the notations would be more complicated.

If a basis $\eta_1, \cdots, \eta_n$ has its elements all multiples of Q except at infinity then the expansions for η_k at the three places over x_0 will have the forms

$$\text{at } A: \quad \eta_k = \alpha_{k0} t^{\lambda} + \cdots + \alpha_{ka-1} t^{\lambda+a-1} + \cdots,$$

$$\text{at } B: \quad \eta_k = \beta_{k0} t^{\mu} + \cdots + \beta_{kb-1} t^{\mu+b-1} + \cdots,$$

$$\text{at } C: \quad \eta_k = \gamma_{k0} t^{\nu} + \cdots + \gamma_{kc-1} t^{\nu+c-1} + \cdots.$$

The determinant

$$\Delta = \begin{vmatrix} \alpha_{10} & \cdots & \alpha_{n0} \\ \cdots & \cdots & \cdots \\ \alpha_{1a-1} & \cdots & \alpha_{na-1} \\ \beta_{10} & \cdots & \beta_{n0} \\ \cdots & \cdots & \cdots \\ \beta_{1b-1} & \cdots & \beta_{nb-1} \\ \gamma_{10} & \cdots & \gamma_{n0} \\ \cdots & \cdots & \cdots \\ \gamma_{1c-1} & \cdots & \gamma_{nc-1} \end{vmatrix}$$

has numerous applications in the following pages and will be referred to always as the determinant Δ for the basis $\eta_1, \cdots, \eta_n$ at the value $x = x_0$. It is clear that an analogous determinant can be constructed for every finite value x_0 no matter how many cycles the algebraic function $y(x)$ may have corresponding to it. We can now prove the following theorem:

THEOREM 2. *A necessary and sufficient condition that a set $\eta_1, \cdots, \eta_n$ of multiples of Q except at $x = \infty$ be a basis for the divisor Q is that at every finite value x_0 their determinant Δ be different from zero.*

To prove the necessity of this condition suppose that the determinant Δ is equal to zero at a value x_0 for a set $\eta_1, \cdots, \eta_n$ of multiples of Q except at infinity. Then there is a set of constants $C_1, \cdots, C_n$ satisfying the linear equations whose coefficients are the rows of Δ, and the numerator of the function

$$r = \frac{C_1 \eta_1 + \cdots + C_n \eta_n}{x - x_0}$$

has orders at least equal to $\lambda + a$, $\mu + b$, $\nu + c$ at the cycles A, B, C, while the denominator has orders a, b, c. Hence η is a multiple of Q except at infinity, not expressible in the form (4), and $\eta_1, \cdots, \eta_n$ can not be a basis for Q.

The condition is also sufficient. For in the first place a set of functions $\eta_1, \cdots, \eta_n$ having the property of the theorem necessarily has its determinant $|\eta_k(x, y_i)|$ not identically zero, because at a value x_0 having n cycles distinct from those of Q the determinant $|\eta_k(x, y_i)|$ is exactly the determinant Δ and therefore different from zero by hypothesis. Every function $\eta(x, y)$ is therefore expressible in terms of $\eta_1, \cdots, \eta_n$ in the form

$$\eta = \frac{g_1(x)\,\eta_1 + \cdots + g_n(x)\,\eta_n}{d(x)}$$

where $g_1(x), \cdots, g_n(x), d(x)$ are polynomials having no common factor. When a function η is a multiple of Q execpt at $x = \infty$ the denominator $d(x)$ must be a constant. For one may readily verify successively that in case $d(x)$ had a factor $(x - x_0)$ the two functions

$$\frac{g_1(x)\,\eta_1 + \cdots + g_n(x)\,\eta_n}{x - x_0}, \qquad \frac{g_1(x_0)\,\eta_1 + \cdots + g_n(x_0)\,\eta_n}{x - x_0}$$

would also be multiples of Q except at infinity. Since the determinant Δ is different from zero at $x = x_0$ the last one would surely have a lower order than that prescribed by Q at one at least of the places over $x = x_0$, which is a contradiction. Hence the denominator $d(x)$ has no factor $x - x_0$ and is constant. Every multiple $\eta(x, y)$ of Q except at infinity is therefore surely expressible in the form (4).

With the help of the last theorem we may proceed to the proof that there is a basis for every divisor Q. In the first place it is evident that a function $\eta(x, y)$ which is not a multiple of Q except at infinity can be made into one by multiplying it by a polynomial in x. For if $\eta(x, y)$ is multiplied by a sufficiently high power of $x - x_0$ the orders of the product at the cycles corresponding to x_0 may be made to exceed those required by Q. It is clear from this remark that a basis $\eta_1, \cdots, \eta_n$ can always be easily made over into one whose functions are all multiples of Q except at infinity.

If a basis $\eta_1, \cdots, \eta_n$ has its functions all multiples of Q except at infinity then the order of the determinant $|\eta_k(x, y_i)|$ at a value x_0 is surely greater than the sum of the orders required by Q at the cycles corresponding to x_0. For if at such a cycle Q requires the expansion of its multiples to begin with a term in $(x - x_0)^{\frac{\mu}{r}}$, then r rows of the determinant $|\eta_k(x, y_i)|$ will have the factor $(x - x_0)^{\frac{\mu}{r}}$ in each element and the determinant itself will have at least the factor $(x - x_0)^{\mu}$. The theorem which we wish to prove with the help of these remarks is now the following one:

THEOREM 3. *For every divisor Q there exists a basis $\eta_1, \cdots, \eta_n$ such that the totality of multiples of Q except at infinity is identical with the totality of functions expressible in the form*

$$\eta(x, y) = g_1(x)\,\eta_1 + \cdots + g_n(x)\,\eta_n$$

where the coefficients $g_k(x)$ are polynomials in x.

To prove this, suppose that $\eta_1, \cdots, \eta_n$ is a basis of functions which are multiples of Q except at infinity. If it is not a basis for Q, there will be a value x_0 at which its determinant Δ vanishes. Let $C_1, \cdots, C_n$ be constants satisfying the linear equations whose coefficients are the rows of Δ, and suppose that C_k is one of them which is different from zero. Then the set of functions

$$\eta_1, \cdots, \eta_{k-1}, \quad \frac{C_1\eta_1 + \cdots + C_n\eta_n}{x - x_0}, \quad \eta_{k+1}, \cdots, \eta_n$$

is also a basis with elements multiples of Q except at infinity, and the determinant of its conjugates is that of the original basis multiplied by $C_k/(x - x_0)$. If the determinant Δ of the new basis still vanishes at x_0, this process may be repeated. It can be repeated a finite number of times only, however, before reaching a basis for which the determinant Δ is different from zero, since after each repetition the order of the determinant $\eta_k(x, y_i)$ at $x = x_0$ is decreased by unity, and we have seen above that for a basis of multiples of Q except at infinity the order of this determinant at $x = x_0$ has a minimum. If the determinant Δ has been made different from zero, as just described, at all the values x_0 corresponding to places in Q, and at all the places where the determinant of the conjugates of the basis originally vanished, then the basis will have $\Delta \neq 0$ at every x_0 and will be a basis for Q. We know that there are only a finite number of values x_0 at which such alterations must be made since Q has only a finite number of factors and the determinant of conjugates only a finite number of zeros.

4. **Multiples of a divisor.** We have seen in § 2 that a basis $\eta_1, \cdots, \eta_n$ for the totality of rational functions $\eta(x, y)$ can be transformed into an equivalent basis $\zeta_1, \cdots, \zeta_n$ by a linear transformation of the form

$$\zeta_k = g_{k1}\eta_1 + \cdots + g_{kn}\eta_n \qquad (k = 1, \cdots, n) \tag{5}$$

in which the coefficients g_{ki} are rational in x and have a determinant not identically zero. A similar relationship can be established for every pair of basis for a divisor Q, as indicated in the following theorem:

THEOREM 4. *If $\eta_1, \cdots, \eta_n$ is a basis for a divisor Q then a necessary and sufficient condition for $\zeta_1, \cdots, \zeta_n$ to be also such a basis is that $\zeta_1, \cdots, \zeta_n$ be expressible in the form* (5) *with coefficients $g_{ki}(x)$ polynomials in x and with a determinant $|g_{ki}|$ equal to a constant different from zero.*

We know that if $\zeta_1, \cdots, \zeta_n$ is to be a basis for Q its functions must be multiples of Q except at $x = \infty$ and hence uniquely expressible in terms of $\eta_1, \cdots, \eta_n$ with polynomial coefficients g_{ki}, since every such multiple is so

expressible. Similarly $\eta_1, \cdots, \eta_n$ must be expressible in terms of $\zeta_1, \cdots, \zeta_n$ with polynomial coefficients h_{kl}. The determinants $|g_{kl}|$ and $|h_{kl}|$ are both polynomials in x and their product is unity. Hence both must be constants. Conversely one may easily see that a set of functions $\zeta_1, \cdots, \zeta_n$ related to $\eta_1, \cdots, \eta_n$ as described in the theorem will surely be a basis for Q, so that the theorem is completely established.

By means of the transformations described in the last theorem bases for Q can be found which have special properties of great assistance in the proofs which we shall discuss in this and the following sections. We may define the *column order* of a function $\eta(x, y)$ at $x = \infty$ as the minimum of the exponents in its n expansions at the cycles for $x = \infty$. If the column orders of the functions of a basis $\eta_1, \cdots, \eta_n$ for Q are the numbers r_k $(k = 1, \cdots, n)$, then the conjugates of these functions will have at $x = \infty$ expansions of the form

$$\eta_k(x, y_i) = C_{ik}\left(\frac{1}{x}\right)^{r_k} + \cdots, \qquad (i, k = 1, \cdots, n), \tag{6}$$

where one at least of each set $C_{1k}, \cdots, C_{nk}$ is different from zero. A basis for Q is said to be *normal at* $x = \infty$ if the determinant $|C_{ik}|$ is different from zero. It is evident that for such a basis the order of the determinant $|\eta_k(x, y_i)|$ at $x = \infty$ is exactly $r_1 + \cdots + r_n$.

THEOREM 5. *For every divisor Q there exists a basis $\eta_1, \cdots, \eta_n$ which is normal at $x = \infty$. When expressed in terms of such a basis a multiple*

$$\eta(x, y) = g_1(x)\,\eta_1 + \cdots + g_n(x)\,\eta_n \tag{7}$$

of Q except at $x = \infty$ has its column order at $x = \infty$ the smallest of the numbers $r_k - \mu_k$ $(k = 1, \cdots, n)$ where r_k is the column order of η_k and μ_k the degree of the polynomial $g_k(x)$.

To establish these statements let us order the functions $\eta_1, \cdots, \eta_n$ of an arbitrarily selected basis for Q so that their column orders satisfy the inequalities $r_1 \geqq r_2 \geqq \cdots \geqq r_n$. If the determinant $|C_{ik}|$ of cofficients from the expansions (6) is equal to zero there will exist constants $C_1, \cdots, C_n$ not all zero satisfying the linear equations whose coefficients are the rows of this determinant. Let C_k be the last one which is different from zero. Then the basis $\eta_1, \cdots, \eta_{k-1}, \eta'_k, \eta_{k+1}, \cdots, \eta_n$, with

$$\eta'_k = C_1 x^{r_1 - r_k}\eta_1 + \cdots + C_{k-1} x^{r_{k-1} - r_k}\eta_{k-1} + C_k \eta_k,$$

is also a basis for Q and has the same column orders except that the order r'_k for η'_k is at least one greater than r_k. If the new basis is not

normal at $x = \infty$ the process can be repeated. It can be repeated only a finite number of times before attaining a normal basis, however, since at each step the sum of the column orders of the basis is increased by at least unity, and this sum is at most equal to the order of the determinant of the conjugates of the basis, which is unchanged by the transformation.

Let μ be the smallest of the numbers $r_k - \mu_k$ so that $r_k - \mu_k \geqq \mu$ $(k = 1, \cdots, n)$. Then the coefficient $g_k(x)$ has degree $\mu_k \leqq r_k - \mu$, and has at $x = \infty$ an expansion of the form

$$g_k(x) = C_k \left(\frac{1}{x}\right)^{\mu - r_k} + \cdots.$$

At least one of the coefficients C_k is different from zero. The expansions of the function (7) at the cycles for $x = \infty$ have therefore the form

$$\eta(x, y_i) = \left(\frac{1}{x}\right)^{\mu} \sum_k C_{ik} C_k + \cdots \qquad (i = 1, \cdots, n),$$

and one at least of the coefficients of $(1/x)^{\mu}$ is necessarily different from zero since the determinant $|C_{ik}|$ is not zero. The column order of η is therefore μ.

So far we have considered only "multiples of a divisor Q except at $x = \infty$", whose usefulness is of an auxiliary sort. The multiples which are of greater importance are those which have orders greater than or equal to the orders prescribed by Q at every cycle whatsoever of the algebraic function $y(x)$, including those at $x = \infty$. It is evident that some divisors will have no multiples, an example being a divisor Q which has positive but no negative exponents. A multiple $\eta(x, y)$ of such a divisor would necessarily have all of its orders greater than or equal to zero, and the sum of the orders of η could not be equal to zero as we know it must be. The following theorem describes the character and the number of the multiples of a divisor Q.

THEOREM 6. *If a divisor Q has a multiple, it has a set $\sigma_1(x, y), \cdots, \sigma_\nu(x, y)$ of linearly independent ones such that every multiple of Q is expressible in the form*

$$\sigma = c_1 \sigma_1 + \cdots + c_\nu \sigma_\nu$$

with constant coefficients. If Q has no cycles at infinity and if $\eta_1, \cdots, \eta_n$ is a basis for Q normal at $x = \infty$ with column orders

$$r_1 \geqq r_2 \geqq \cdots \geqq r_s \geqq 0 > r_{s+1} \geqq \cdots \geqq r_n, \tag{8}$$

then the number ν of such multiples is

$$\nu = (r_1+1)+\cdots+(r_s+1).$$

To prove the theorem we suppose that a transformation $x' = 1/(x-x_0)$ has been applied so that none of the cycles of Q are at infinity. When $\eta_1, \cdots, \eta_n$ is a basis with the properties presupposed in the theorem the function

$$\eta = g_1(x)\eta_1+\cdots+g_n(x)\eta_n$$

will be a multiple of Q at all cycles, including those at $x=\infty$, if and only if its column order at $x=\infty$ is greater than or equal to zero. According to the last theorem this can never be so if all of the column orders r_k are negative, since then all the numbers $r_k-\mu_k$ are negative, and in this case the divisor Q has no multiples. When some of the column orders are positive, as indicated in the arrangement (8), the function η will have its column order greater than or equal to zero at $x=\infty$, according to Theorem 5, if and only if it is expressible in the form

$$\eta = g_1(x)\eta_1+\cdots+g_s(x)\eta_s \tag{9}$$

with $r_k-\mu_k \geqq 0$ for each coefficient $g_k(x)$ $(k=1,\cdots,s)$. It is clear from this remark that the degree μ_k of each $g_k(x)$ can be r_k but no greater. The functions

$$\eta_1,\ x\eta_1,\ \cdots,\ x^{r_1}\eta_1,\ \cdots,\ \eta_s,\ x\eta_s,\ \cdots,\ x\eta^{r_s}$$

therefore constitute a set of multiples of Q in terms of which all such multiples are expressible linearly with constant coefficients, and their number ν is that given in the theorem. They are linearly independent since no linear expression of the form (9) can vanish identically when the determinant $|\eta_k(x,y_i)|$ is different from zero. This completes the proof of the theorem. We can infer in a similar manner the truth of the following useful corollary.

COROLLARY. *Let D be the divisor which is the product of the cycles at $x=\infty$, and let Q be a divisor having no cycles at $x=\infty$. If $\eta_1,\cdots,\eta_n$ is a basis for Q normal at $x=\infty$ and having there the column orders indicated in the arrangement* (8) *then the number of multiples of the divisor DQ is $\nu = r_1+\cdots+r_s$. Furthermore if*

$$r_1 \geqq r_2 \geqq \cdots \geqq r_t \geqq 2 > r_{t+1} \geqq \cdots \geqq r_n,$$

then the number of multiples of D^2Q is

$$\nu = (r_1 - 1) + \cdots + (r_t - 1). \tag{10}$$

The proof of the first part of the corollary is like that of the theorem except that the degree of each $g_k(x)$ in the expression (9) can now not exceed $r_k - 1$ if we wish $\eta(x, y)$ to have a zero of order one at least at each cycle for $x = \infty$.

Similarly the multiples of D^2Q are the multiples of Q which have zeros of order two at least at the cycles for $x = \infty$. By an argument similar to the one just made it follows that these multiples are the functions

$$\eta = g_1(x)\,\eta_1 + \cdots + g_t(x)\,\eta_t$$

for which each polynomial $g_k(x)$ $(k = 1, \cdots, t)$ has degree at most equal to $r_k - 2$. Hence their number is the number ν of the corollary.

It is not always easy to compute the number of linearly independent multiples of a divisor from the criteria given in the theorem above and its corollary, but we shall see that in a number of important cases this computation can be readily made with the help of the next theorem. Let the values of x corresponding to the cycles of a divisor $Q = P_1^{\mu_1} \cdots P_s^{\mu_s}$ be the finite values $x_1, \cdots, x_s$. Then the *ideal norm of the divisor* Q is defined to be the product

$$N(Q) = (x - x_1)^{\mu_1} \cdots (x - x_s)^{\mu_s}.$$

If a divisor has cycles at $x = \infty$ they are neglected in forming the ideal norm, but otherwise the definition is the same.

We have seen in § 1 that there are only a finite number of branch cycles for the algebraic function $y(x)$. The *divisor of the branch cycles* is defined to be the divisor $X = \prod P^{r-1}$, where P is a branch cycle and r the number of roots y_i of the equation $f(x, y) = 0$ furnished by it, and the product is taken for all of the branch cycles. With the help of these notations the theorem which will now be proved is as follows:

THEOREM 7. *For every basis $\eta_1, \cdots, \eta_n$ of a divisor Q we have*

$$|\,\eta_k(x, y_i)\,|^2 = cN(Q)^2 N(X)$$

where c is a constant factor.

It is clear that the power of $x - x_0$ which occurs in the determinant $|\,\eta_k(x, y_i)\,|$ is the same for all bases of Q since for two equivalent bases the values of this determinant differ only by a constant factor (Theorem 4).

Let us suppose therefore that the basis $\eta_1, \cdots, \eta_k$ has already been prepared by a linear transformation (5) with constant coefficients g_{kl} so that the determinant Δ for $\eta_1, \cdots, \eta_k$ at $x = x_0$ is the identity determinant. For the illustrative case used above, in which there are three cycles corresponding to $x = x_0$, the following table indicates the exponents of the lowest powers of $(x - x_0)$ in the expansions of the elements of the determinant $\eta_k(x, y_i)$ at the three cycles.

	η_1 η_2 $\cdots$ η_a	η_{a+1} $\cdots$ η_{a+b}	η_{a+b+1} $\cdots$ η_n
$\eta(x, y_1)$	$\frac{\lambda}{a}$ $\frac{\lambda+1}{a}$ $\cdots$ $\frac{\lambda+a-1}{a}$		
$\cdots$	$\cdots$ $\cdots$ $\cdots$ $\cdots$	$\geqq \frac{\lambda}{a}+1$	$\geqq \frac{\lambda}{a}+1$
$\eta(x, y_a)$	$\frac{\lambda}{a}$ $\frac{\lambda+1}{a}$ $\cdots$ $\frac{\lambda+a-1}{a}$		
$\eta(x, y_{a+1})$		$\frac{\mu}{b}$ $\cdots$ $\frac{\mu+b-1}{b}$	
$\cdots$	$\geqq \frac{\mu}{b}+1$	$\cdots$ $\cdots$ $\cdots$	$\geqq \frac{\mu}{b}+1$
$\eta(x, y_{a+b})$		$\frac{\mu}{b}$ $\cdots$ $\frac{\mu+b-1}{b}$	
$\eta(x, y_{a+b+1})$			$\frac{\nu}{c}$ $\cdots$ $\frac{\nu+c-1}{c}$
$\cdots$	$\geqq \frac{\nu}{c}+1$	$\geqq \frac{\nu}{c}+1$	$\cdots$ $\cdots$ $\cdots$
$\eta(x, y_n)$			$\frac{\nu}{c}$ $\cdots$ $\frac{\nu+c-1}{c}$

Out of each of the first a rows we may take the factor $(x - x_0)^{\frac{\lambda}{a}}$, and out of the $a - 1$ columns following the first the factor $(x - x_0)$ raised to the power $1/a + \cdots + (a-1)/a$. Hence in all we have from these rows and colums the factor $(x - x_0)$ raised to the power $\lambda + (a-1)/2$. A similar process applied to the two remaining principal minors indicated in the diagram gives for the determinant $|\eta_k(x, y_i)|^2$ the factor $(x - x_0)$ raised to the power

$$2\lambda + 2\mu + 2\nu + (a-1) + (b-1) + (c-1),$$

which is exactly the power of $(x - x_0)$ occuring in the product $N(Q)^2 N(X)$. The same method or proof applies when there are more or fewer than three cycles at $x = x_0$.

If we can show that, after the power of $(x - x_0)$ described above has been removed, the constant term in the expansion for $|\eta_k(x, y_i)|$ is different from zero, we shall have proved our theorem, for then the zeros and poles in the finite x-plane of the two rational functions $|\eta_k(x, y_i)|^2$ and $N(Q)^2 N(X)$ are identical, and these two can differ only by a constant factor. When we set $x = x_0$ in the determinant Δ deprived of the factors $(x - x_0)$ as described, the three principal minors in the squares indicated in the diagram are the only ones which remain. The first one of these, for example, is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \omega^{\lambda} & \omega^{\lambda+1} & \cdots & \omega^{\lambda+a-1} \\ \cdots & \cdots & \cdots & \cdots \\ \omega^{(a-1)\lambda} & \omega^{(a-1)(\lambda+1)} & \cdots & \omega^{(a-1)(\lambda+a-1)} \end{vmatrix}$$

where ω is a primitive a-th root of unity, and it is different from zero since no two of the roots $\omega^{\lambda+k}$ $(k = 0, 1, \cdots, a-1)$ are equal. A similar argument applies to the other minors, and the theorem is therefore proved.

As an immediate consequence of the preceding theorems we have

COROLLARY 1. *For every basis $\eta_1, \cdots, \eta_n$ of a divisor Q normal at $x = \infty$ the eqation*

$$r_1 + \cdots + r_n + q + \frac{w}{2} = 0 \tag{11}$$

is true, where the integers r_k are the column orders of the basis at $x = \infty$, q is the order of the divisor Q, and w the order of the divisor X of the branch places.

To prove the corollary we note first that the order at $x = \infty$ of the determinant $|\eta_k(x, y_i)|^2$ for a basis normal at infinity is $2(r_1 + \cdots + r_n)$, as we have seen above on page 106. According to the last theorem the sum of the orders of $|\eta_k(x, y_i)|^2$ at finite values of x is the same as that of the product $N(Q)^2 N(X)$ which we know to be $2q + w$. The equation (11) then expresses the known fact that the sum of all the orders of the rational function $|\eta_k(x, y_i)|^2$ is zero.

COROLLARY 2. *The number*

$$p = \frac{w}{2} - n + 1,$$

which is called the genus of the algebraic function defined by the equation $f(x, y) = 0$, is always a positive integer or zero.

It is evident from the equation (11) that $w/2$ is an integer and hence that the number p in the last corollary is an integer. To prove that it

is not negative consider a basis $\eta_1, \cdots, \eta_n$ normal at $x = \infty$ for the particular divisor $Q = 1$. This basis must have all of its column orders $r_1, \cdots, r_n$ at $x = \infty$ zero or negative since no rational function can have positive orders at all cycles at $x = \infty$ and no negative ones elsewhere. One at least of the column orders must be zero since otherwise the column order of the function

$$\eta = g_1(x)\,\eta_1 + \cdots + g_n(x)\,\eta_n$$

would always be negative (Theorem 5), and this is impossible since the function $\eta(x, y) =$ constant is certainly a multiple of the divisor $Q = 1$. On the other hand, two of the numbers $r_1, \cdots, r_n$ could not be zero since then both of the corresponding functions of the bases would have no singularities and would be constants, and they would not be linearly independent. Since now $r_1 = 0$ while all other integers $r_2, \cdots, r_n$ are negative, it follows with the help of equation (11), since the order of the divisor $Q = 1$ is $q = 0$, that

$$p = \frac{w}{2} - n + 1 = -r_1 - \cdots - r_n - n + 1 \geqq 0.$$

5. **Complementary bases.** A basis $\zeta_1, \cdots, \zeta_n$ is said to be *complementary* to the basis $\eta_1, \cdots, \eta_n$ if the traces of the products of the functions composing the two bases satisfy the relations

$$T(\eta_k\,\zeta_k) = 1, \qquad T(\eta_k\,\zeta_1) = 0 \qquad (k \neq 1). \tag{12}$$

We have seen in § 2 that the coefficients R_k for the function

$$\zeta = R_1\,\eta_1 + \cdots + R_n\,\eta_n$$

will be uniquely determined when the traces

$$T(\eta_i\,\zeta) = R_1\,T(\eta_i\,\eta_1) + \cdots + R_n\,T(\eta_i\,\eta_n) \quad (i = 1, \cdots, n)$$

are assigned. It follows readily that the functions ζ_k of a basis complementary to $\eta_1, \cdots, \eta_n$ are uniquely determined, and that the relation between the two bases is a reciprocal one.

THEOREM 8. *If a basis $\eta_1, \cdots, \eta_n$ is normal at $x = \infty$ with column orders $r_1, \cdots, r_n$, then its complementary basis $\zeta_1, \cdots, \zeta_n$ is also normal at $x = \infty$ and has the column orders $-r_1, \cdots, -r_n$.*

Let us denote the column orders which are to be determined for the basis $\zeta_1, \cdots, \zeta_n$, by $s_1, \cdots, s_n$, and let d_{ik} be the matrix of coefficients for this basis corresponding to the matrix C_{ik} in equation (6) for the original basis. Then at $x = \infty$ we have the expansion

$$T(\eta_i \zeta_k) = \left(\frac{1}{x}\right)^{r_i+s_k} \sum_{j=1}^{n} C_{ji} d_{jk} + \cdots.$$

These expansions must vanish identically when $i \neq k$ and be identically equal to 1 when $i = k$, on account of the relations (12). Since the determinant $|C_{ik}|$ is different from zero, and since the constants d_{jk} $(j = 1, \cdots, n)$ for a fixed k are not all zero, it follows that the relations

$$\sum_{j=1}^{n} C_{ji} d_{jk} = 0 \quad (i \neq k), \qquad \sum_{j=1}^{n} C_{jk} d_{jk} = 1, \qquad r_k + s_k = 0$$

must hold. We see that $s_k = -r_k$, and that the determinant $|d_{jk}|$ is different from zero since it is, except for interchange of rows and columns, the reciprocal of $|C_{ik}|$.

THEOREM 9. *If $\eta_1, \cdots, \eta_n$ is a basis for a divisor Q, then its complementary basis $\zeta_1, \cdots, \zeta_n$ is a basis for the divisor R defined by the equation $QRX = 1$, in which X is the divisor of the branch places.*

To prove this consider again the illustrative case of a value $x = x_0$ having three cycles A, B, C providing, respectively, a, b, c roots of $f(x, y) = 0$. The argument to be made would be quite similar if there were more or fewer than three places. At the three cycles A, B, C, respectively, let λ, μ, ν be the orders of Q and λ', μ', ν' the minima of the orders of the functions of the complementary basis $\zeta_1, \cdots, \zeta_n$. We suppose the notations for the cycles A, B, C chosen so that

$$(13) \qquad (\lambda + \lambda')/a \leqq (\mu + \mu')/b \leqq (\nu + \nu')/c.$$

Consider now two functions

$$\eta = u_1 \eta_1 + \cdots + u_n \eta_n, \qquad \zeta = v_1 \zeta_1 + \cdots + v_n \zeta_n,$$

for which the coefficients u_i, v_i are constants. Since the determinant Δ on page 103 for the basis $\eta_1, \cdots, \eta_n$ is different from zero at the value x_0 it follows that when a number h of the set $0, 1, \cdots, a-1$ has been selected arbitrarily, the coefficients u_k can be determined so that the function η has at A, B, C, respectively, the expansions

$$\tau = (x-x_0)^{\frac{\lambda+h}{a}} + \alpha(x-x_0)^{\frac{\lambda}{a}+1} + \cdots,$$

$$\eta = \beta(x-x_0)^{\frac{\mu}{b}+1} + \cdots,$$

$$\eta = \gamma(x-x_0)^{\frac{\nu}{c}+1} + \cdots.$$

The coefficient v_k can then be in turn selected so that ζ has orders exactly λ', μ', ν' at A, B, C, respectively, and so that

$$T(\eta\zeta) = u_1 v_1 + \cdots + u_n v_n \neq 0. \tag{14}$$

When the coefficients u_k, v_k have been chosen in this manner, the smallest exponent of $(x-x_0)$ in the expansions of the terms of $T(\eta\zeta)$ is seen with the help of the inequalities (13) to be $(\lambda+\lambda'+h)/a$. But the trace $T(\eta\zeta)$ is a constant different from zero, so that according to the remarks in the last paragraph of § 1 we must have the exponent $(\lambda+\lambda'+h)/a$ less than zero when it is a fraction and equal to zero when it is an inreger. Since there is one integer only in the set of numbers $(\lambda+\lambda'+h)/a$ for $h = 0, 1, \cdots, a-1$ it follows that the largest number of this set must be zero, and therefore that the first of the relations

$$\lambda+\lambda'+a-1 = 0, \quad \mu+\mu'+b-1 = 0, \quad \nu+\nu'+c-1 = 0, \tag{15}$$

is true. A similar argument for the cycles B, C justifies the last two if at each step we use the relations (15) already found, and (13), and note that according to the last paragraph of § 1 the smallest exponent in the terms of $T(\eta\zeta)$ must be $\leqq 0$. From the equations (15) we see therefore that the minimium orders λ', μ', ν' for the basis $\zeta_1, \cdots, \zeta_n$ at the cycles A, B, C for $x = x_0$ are exactly the orders $-\lambda-a+1$, $-\mu-b+1$, $-\nu-c+1$ prescribed by the divisor $R = 1/QX$.

We can prove that at the value x_0 the determinant Δ on page 103 formed with respect to the divisor R for the basis $\zeta_1, \cdots, \zeta_n$ can not be zero. For if it had the value zero, constants v_k, not all zero, could be selected so that at A, B, C the function ζ has, respectively, the expansions

$$\zeta = \alpha'(x-x_0)^{\frac{\lambda'}{a}+1} + \cdots,$$

$$\zeta = \beta'(x-x_0)^{\frac{\mu'}{b}+1} + \cdots,$$

$$\zeta = \gamma'(x-x_0)^{\frac{\nu'}{c}+1} + \cdots.$$

Since for every choice of the constants u_k the orders of η at the cycles A, B, C are at least λ, μ, ν, the exponents of the terms in $T(\eta\zeta)$ would then all be greater than zero, on account of the relations (15). The trace (14) would be zero for every choice of the constants u_k, which is impossible.

Since the above reasoning applies at every finite value x_0 it follows that $\zeta_1, \cdots, \zeta_n$ is a basis for the divisor R, as stated in the theorem.

As a first application of the properties of the complementary basis we may establish formulas for the number of multiples of divisors of the form $D^2/(P_1 \cdots P_\mu X)$ which will be of service in a later section. Let $\eta_1, \cdots, \eta_n$ be a basis for the divisor $Q = 1/(P_1 \cdots P_\mu X)$, normal at $x = \infty$ and with column orders

$$r_1 \geqq r_2 \geqq \cdots \geqq r_t \geqq 2 > r_{t+1} \geqq \cdots \geqq r_n. \tag{16}$$

Since the order of Q is $q = -(\mu + w)$, the formula (11) of page 111 gives the relation

$$r_1 + \cdots + r_n - (\mu + w) + \frac{w}{2} = 0. \tag{17}$$

The basis $\zeta_1, \cdots, \zeta_n$ complementary to $\eta_1, \cdots, \eta_n$ is a basis for the divisor $R = 1/XQ = P_1 \cdots P_\mu$ and has at $x = \infty$ the column orders $-r_1, \cdots, -r_n$. Each of the functions ζ_k must have a negative column order at $x = \infty$ since each has zeros at the cycles $P_1, \cdots, P_\mu$ and must therefore have some poles at infinite places which are the only places where such poles are possible. The numbers r_k are therefore all positive, and those following r_t in the arrangement (16) are unity. Formula (17) with the equation (10) of page 109 therefore give

$$\nu = (r_1 - 1) + \cdots + (r_n - 1) = \frac{w}{2} - n + \mu = p + \mu - 1$$

as the number of multiples of $D^2/(P_1 \cdots P_\mu X)$. When no cycles $P_1, \cdots, P_\mu$ are present, we have $R = 1$ and it follows from the argument at the top of page 112 that $r_n = 0$, while the other column orders are positive. Hence we now have, from formulas (10) and (17) with $\mu = 0$,

$$\nu = (r_1 - 1) + \cdots + (r_n - 1) + 1 = \frac{w}{2} - n + 1 = p$$

as the number of multiples of D^2/X. This proves the following theorem:

THEOREM 10. *Let D be the divisor whose factors are the cycles at $x = \infty$, X the divisor of the branch cycles, and $P_1, \cdots, P_\mu$ arbitrarily chosen cycles.*

Then the number of linearly independent multiples of the divisor D^ν/X is exactly $\nu = p$, and the number for $D^\nu/(P_1 \cdots P_\mu X)$ $(\mu \geqq 1)$ is

$$\nu = p + \mu - 1.$$

6. The invariant property of the genus number. If the poles and zeros of a rational function $\eta(x, y)$ are at the cycles $P_1, \cdots, P_s$ and have the orders $\mu_1, \cdots, \mu_s$, then the divisor $Q_\eta = P_1^{\mu_1}, \cdots, P_s^{\mu_s}$ is called *the divisor of* η. Since the sum of the orders of a rational function is always zero, it follows that the order of the divisor Q_η must be zero. Conversely one might expect that there would be a rational function corresponding in this way to every divisor Q of order zero, but this is not always the case. It is true in the case of functions of a single variable x which may be regarded as rational functions of the algebraic function defined by the equation $y - x = 0$. For we can easily construct out of factors of the type $|x - x_0|^\lambda$ a rational function of x with arbitrarily prescribed poles and zeros in the complex x-plane, provided only that the sum of the orders of these poles and zeros is zero. But for rational functions $\eta(x, y)$ of a more general algebraic function no such simple construction is possible, and it may be that for some divisors of order zero no corresponding rational function exists.

Some of the factors in Q_η have positive exponents and some negative. We may agree to denote the product of those with positive exponents by N_η, and may express the divisor in the form $Q_\eta = N_\eta / D_\eta$, where the meaning of D_η is evident. At every cycle P of a factor P^a of the denominator D_η the expansions (3) of page 98 for the algebraic function and its derivative have the forms

$$\eta = \frac{\beta}{t^a} + \cdots, \qquad \frac{d\eta}{dt} = -a\beta \frac{t^{a-1}}{t^{2a}} + \cdots, \tag{18}$$

where β is a constant different from zero. At other cycles these expansions have the form

$$\eta = \alpha + \beta t^a + \cdots, \qquad \frac{d\eta}{dt} = a\beta t^{a-1} + \cdots, \tag{19}$$

where β is again different from zero and α a constant which vanishes at the factors of N_η but not elsewhere. A cycle P of either type is called a *branch cycle for* $\eta(x, y)$ if $a \neq 1$. The *divisor of the branch cycles for* η is defined to be $X_\eta = \prod P^{a-1}$, where the product is taken for all the branch cycles P of η. We may for convenience denote the orders of X_η and D_η by

w_η and n_η, respectively. For the function $\eta = x$, the notations which have just been introduced give the special cases X_x, D_x, w_x, n_x which have been denoted in the preceding pages by X, D, w, n.

THEOREM 11. *The divisor whose orders are identical with those of the derivative $d\eta/dt$ at the various cycles of an algebraic function is X_η/D_η^2. The order $w_\eta - 2n_\eta$ of this divisor is the same for all rational functions η.*

The first part of the theorem is evident after an examination of the exponents in the expressions (18) and (19) for $d\eta/dt$. To prove the second part we consider a second rational function $\xi(x, y)$ and notice that, by elimination of x and y from the three equations

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y), \qquad f(x, y) = 0,$$

a relation $\varphi(\xi, \eta) = 0$ can always be found in which $\varphi(\xi, \eta)$ is a polynomial. Since all of the expansions for ξ and η satisfy this equation identically it follows that

$$\frac{d\eta}{dt} = -\left(\frac{\partial\varphi}{\partial\xi} \Big/ \frac{\partial\varphi}{\partial\eta}\right)\frac{d\xi}{dt},$$

which shows that at every cycle the order of $d\eta/dt$ is the sum of the order of a rational function of x, y and the order of $d\xi/dt$. The divisor of $d\eta/dt$ is therefore the product of the divisor of $d\xi/dt$ by that of a rational function, and the orders of the divisors of $d\eta/dt$ and $d\xi/dt$ are therefore always equal.

As a result of the invariance of the expression $w_\eta - 2n_\eta$, we see that the genus of our algebraic function is also expressible in the form

$$p = \frac{1}{2}w_\eta - n_\eta + 1,$$

since, according to Theorem 11, this number will have the same value no matter what rational function $\eta(x, y)$ is used in its computation. From Corollary 2 of page 111 we know that p is always a positive integer or zero and hence that $w_\eta/2$ is always an integer.

7. **Construction of elementary integrals.** Integrals of the form

$$I = \int \eta(x, y)\, dx,$$

where $\eta(x, y)$ is a rational function of x and y, are the so-called Abelian integrals associated with the algebraic equation $f(x, y) = 0$. They play

a most important and interesting role in the theory of the algebraic functions defined by such equations. If we write such an integral in the form

$$I = \int \eta(x, y) \frac{dx}{dt} dt$$

and substitute the expansions of η and dx/dt, it is evident that at each cycle the value of the integral will be expressible in terms of t by means of a series of the form

$$\frac{A_{-p}}{t^p} + \cdots + \frac{A_{-1}}{t} + A \log t + A_0 + A_1 t + \cdots. \tag{20}$$

If the series has no logarithmic term or terms with negative exponents, the cycle is called an ordinary cycle for the integral; if negative terms are present but no logarithmic term, it is a pole; if a logarithmic term is present, it is a logarithmic singularity. Integrals which have no singularities are called integrals of the first kind; those which have poles but no logarithmic singularities are of the second kind; while those with logarithmic singularities are of the third kind.

All integrals of the first kind may be thought of as elementary integrals, but an integral of the second kind is called an elementary integral only when it has no singularity except a single pole with an expansion of the form

$$\frac{1}{t^\mu} + A_0 + A_1 t + \cdots.$$

An integral of the third kind is an elementary integral if it has only two singularities at which its expansions are

$$\log t + A_0 + A_1 t + \cdots, \qquad -\log t + B_0 + B_1 t + \cdots.$$

The coefficients A in the expansion (20) are the residues of the function $\eta(x, y)$ and we have found on page 100 that for every such function the sum of the residues is zero. We see then that the elementary integral of the third kind is as simple as one could hope to find, since every such integral must have at least two logarithmic singularities.

We shall see presently that every Abelian integral whatsoever is expressible as a sum of elementary integrals of the first, second, and third kinds multiplied by constant coefficients, but for this result to have significance we must be certain that elementary integrals of three kinds actually exist. For integrals I of the first kind it is evident that at every

cycle the expansion for $\eta\,dx/dt$ must be without terms in negative powers of t, and the sum of the orders of η and dx/dt must therefore be positive or zero. This means that η is a multiple of the reciprocal of the divisor of dx/dt, and according to Theorem 11 of page 117 this reciprocal is D^2/X. From Theorem 10 of page 115 the number of linearly independent multiples of this divisor is p, and we have the following theorem.

THEOREM 12. *The number of linearly independent integrals of the first kind is exactly the genus p. If $I_1, \cdots, I_p$ are such integrals of the first kind then every other integral of the first kind is expressible uniquely with constant coefficients in the form*

$$I = c_1 I_1 + \cdots + c_p I_p + c.$$

For an elementary integral of the second kind with no singularity except a simple pole at a cycle P the product $\eta\,dx/dt$ can have no singularity except a pole of order two at P. It follows readily that the number of linearly independent functions η for which the product $\eta\,dx/dt$ has these properties is the same as the number of multiples of the divisor D^2/P^2X. According to Theorem 10 of page 115 this number is $\nu = p+1$. One at least of the multiples η must give the product $\eta\,dx/dt$ a negative order at P since otherwise all would be integrands of integrals of the first kind, and only p of these are linearly independent. For an η which provides a negative order the expansion at P must have the form

$$\eta\frac{dx}{dt} = \frac{A_{-2}}{t^2} + A_0 + A_1 t + \cdots, \qquad (A_{-2} \neq 0),$$

the term in $1/t$ being absent since the sum of the residues of η is zero. The integrand $-\eta/A_{-2}$ gives the integral of the following theorem for the case when $\mu = 1$.

THEOREM 13. *For every cycle P there exists an elementary integral $J_\mu(P)$ of the second kind with an expansion at P of the form*

$$J_\mu(P) = \frac{1}{t^\mu} + A_0 + A_1 t + \cdots.$$

The integrals

$$J_\mu(P) + c_1 I_1 + \cdots + c_p I_p + c$$

are also of this type and there are no others.

The last statement is evident since the difference of two integrals $J_\mu(P)$ with the same μ is necessarily an integral of the first kind.

The proof for integers $\mu > 1$ can be made by an induction. An integrand function η furnishing an integral $J_\mu(P)$ must give $\eta\, dx/dt$ no singularities except a pole of order $\mu+1$ at the cycle P. The number of linearly independent functions η providing only such a pole for $\eta\, dx/dt$ is the number $\nu = p+\mu$ (Theorem 10, page 115) of linearly independent multiples of the divisor $D^{\mu}/(P^{\mu+1}X)$. If Theorem 13 is true for all integers $\leqq \mu-1$, there can be only $p+\mu-1$ linearly independent η's giving $\eta\, dx/dt$ at most a pole of order μ at P, and among the $p+\mu$ giving poles of order $\mu+1$ at most there must be an η giving order exactly $\mu+1$. By subtracting suitable constant multiples of $J_1(P), \cdots, J_{\mu-1}(P)$ from the integral with η as its integrand, and multiplying finally by a constant if necessary, an integral $J_\mu(P)$ can be constructed with the properties of the theorem.

THEOREM 14. *For an arbitrary pair of distinct cycles P_1, P_2 there exists an elementary integral of the third kind $K(P_1, P_2)$ with expansions at P_1 and P_2, respectively, of the forms*

$$+\log t + A_0 + A_1 t + \cdots, \qquad -\log t + B_0 + B_1 t + \cdots.$$

The integrals

$$K(P_1, P_2) + c_1 I_1 + \cdots c_p I_p + c$$

are also of this type and there are no others.

To construct such an integral $K(P_1, P_2)$ one must find an integrand function η which gives $\eta dx/dt$ no singularities except simple poles at P_1 and P_2, and which is therefore a multiple of the divisor D^2/P_1P_2X. According to Theorem 10 of page 115 there are $\nu = p+1$ linearly independent multiples of this divisor, one at least of which must give $\eta dx/dt$ a pole at one at least of the cycles P_1, P_2. But if η gives $\eta dx/dt$ a simple pole at P_1 with an expansion

$$\frac{A}{t} + A_0 + A_1 + \cdots,$$

it must also provide a simple pole at P_2 with an expansion

$$-\frac{A}{t} + B_0 + B_1 t + \cdots,$$

since the sum of the residues of η is zero. The integrand η/A furnishes the integral $K(P_1, P_2)$ of the theorem.

THEOREM 15. *Every Abelian integral I is expressible linearly in terms of elementary integrals with constant coefficients.*

Let L be a cycle at which the integral I has a logarithmic singularity and at which the residue of the integrand function η of I is denoted by A. We may select arbitrarily a cycle L_0 distinct from all the singularities L. The difference

$$I-\sum_L AK(L, L_0),$$

where the sum is taken for all the logarithmic singularities L, has no logarithmic singularities at the cycles L, and also none at L_0 since the sum of the residues A is zero. It may still have poles, however, at cycles P with expansions of the form

$$\frac{B_{-\mu}}{t^\mu}+\cdots+\frac{B_{-1}}{t}+B_0+B_1 t+\cdots.$$

The difference

$$I-\sum_L AK(L, L_0)-\sum_P [B_{-\mu}J_\mu(P)+\cdots+B_{-1}J_1(P)],$$

where the last sum is taken for all the poles P, is an integral of the first kind linearly expressible as indicated in Theorem 12 of page 119.

8. **The Riemann-Roch Theorem.** The theorem to be proved in this section is a famous one in the theory of algebraic functions which has many applications in both geometry and analysis. It is not the purpose of these pages to discuss these applications, but rather to present a proof of the theorem free from restrictive assumptions upon the singularities of the algebraic equation defined by the equation $f = 0$, and based upon the theory of multiples of a divisor. For this purpose we need the following two lemmas.

LEMMA 1. *If Q is an arbitrarily selected divisor, and Q_ξ the divisor of a rational function $\xi(x, y)$, then the number of linearly independent multiples of Q is the same as that for $Q_\xi Q$.*

This is evident because, if σ is a multiple of Q, then the product $\xi\sigma$ is a multiple of $Q_\xi Q$ and conversely.

LEMMA 2. *If $P_1, \cdots, P_m$ are cycles selected arbitrarily among those of a divisor Q, there is always a rational function η such that the divisor $Q_\eta Q$ contains none of the places $P_1, \cdots, P_m$.*

We can always make a transformation $x' = 1/(x-x_0)$ such that after the transformation none of the cycles $P_1, \cdots, P_m$ i at $x' = \infty$. Suppose that this has already been done so that none of them is at $x = \infty$, and let $\mu_1, \cdots, \mu_m$ be the exponents which they have in Q. The divisor $Q_1 = P_1^{-\mu_1} \cdots P_m^{-\mu_m}$ has

a basis $\eta_1, \cdots, \eta_n$ for which the determinant Δ of page 103 is different from zero at every finite value x_0. If we select the constant coefficients in the expression

$$\eta = c_1\eta_1 + \cdots + c_n\eta_n$$

so that they do not satisfy the linear equations whose coefficients are the rows of the determinants Δ corresponding to the values x_0 for the cycles $P_1, \cdots, P_m$, then η will have exactly the orders $-\mu_1, \cdots, -\mu_m$ at these cycles, and the divisor $Q_\eta Q$ will contain none of the cycles.

We are now ready to prove the following

THEOREM 16. (Riemann-Roch.) *If Q and Q' are two divisors such that $QQ' = Q_\xi D^2/X$, where Q_ξ is the divisor of a rational function $\xi(x, y)$ and X/D^2 the divisor of dx/dt, then the numbers ν, ν' of linearly independent multiples of Q, Q' and the orders q, q' of these divisors satisfy the relations*

$$\nu = \nu' - q - p + 1, \qquad q + q' + 2p - 2 = 0,$$

which may also be written in the form

$$2\nu + q = 2\nu' + q', \qquad q + q' + 2p - 2 = 0.$$

Let us first of all make a transformation $x' = 1/(x - x_0)$, if necessary, in order to remove the cycles of the divisor X from $x = \infty$. The relation between Q and Q' then becomes

$$QQ' = Q_\xi \frac{D^2}{D_{x'}^2} \frac{D_{x'}^2}{X} = Q_{\xi'} \frac{D_{x'}^2}{X_{x'}},$$

since $Q_\xi D^2/D_{x'}^2$ is readily seen to be the divisor of the rational function $\xi' = \xi/(x - x_0)^2$, and since the branch cycles in X are transformed into those of $X_{x'}$.

Suppose that this transformation has already been made so that none of the cycles of X is at $x = \infty$. We may write the divisor equation of the theorem in the form

$$Q_1 Q_1' = \frac{1}{X}, \qquad Q_1 = \frac{Q_\eta Q}{Q_\xi D}, \qquad Q_1' = \frac{Q'}{Q_\eta D},$$

and select the rational function η (Lemma 2) so that the divisor Q_1 contains no cycle at $x = \infty$. The divisor Q_1' then also has this property since X has no cycles at $x = \infty$. The numbers of linearly independent multiples

of Q and Q' are now respectively equal (Lemma 1) to those numbers for DQ_1 and DQ_1'.

Let $\eta_1, \cdots, \eta_n$ be a basis for the divisor Q_1 with column orders arranged in the order (8) of page 107 so that the number of multiples of DQ_1 is given by the formula $\nu = r_1 + \cdots + r_s$ of the corollary on page 108. The complementary basis $\zeta_1, \cdots, \zeta_n$ is a basis for the divisor Q_1' with column orders

$$-r_1 \leqq -r_2 \leqq \cdots \leqq -r_s \leqq 0 < -r_{s+1} \leqq \cdots \leqq -r_n$$

(Theorem 8), and the number of multiples of DQ_1' (Corollary, p. 108) is

$$\nu' = -r_{s+1} - \cdots - r_n.$$

Hence by formula (11) of page 111 we have

$$(23) \qquad \nu - \nu' = r_1 + \cdots + r_n = -(q-n) - \frac{w}{2} = -q - p + 1,$$

since the order of Q_1 is $q-n$ and $p = (w/2) - n + 1$. Furthermore, the equation $QQ' = Q_\xi D^2/X$ shows that

$$(24) \qquad q + q' = 2n - w = 2 - 2p.$$

The relations (23) and (24) are the first pair of the theorem which can easily be transformed into the second.

The theorem can be given the following slightly different form from which some of the properties of rational functions with prescribed poles can be more readily derived:

COROLLARY 1. *If Q and Q' are two divisors such that $QQ' = Q_\xi X/D^2$, then the orders q, q' of these divisors and the numbers μ, μ' of linearly independent multiples of $1/Q$ and $1/Q'$ satisfy the relations*

$$(25) \qquad \mu = \mu' + q - p + 1, \qquad q + q' = 2p - 2,$$

which may also be written in the form

$$2\mu - q = 2\mu' - q', \qquad q + q' = 2p - 2.$$

It is evident that a divisor whose order is positive can have no multiple since the number of zeros of such a multiple would necessarily exceed the

number of its poles, whereas, as we know, these numbers must be the same. If $q>2p-2$, then the second formula (25) shows that q' is negative and the number μ' of multiples of $1/Q'$ is zero. Consequently we have

COROLLARY 2. *If Q is a divisor of order $q>2p-2$, then the number μ of multiples of $1/Q$ is exactly $\mu = q-p+1$.*

If $P_1, \cdots, P_s$ are arbitrarily selected cycles of our algebraic function, and $\mu_1, \cdots, \mu_s$ an arbitrarily selected set of positive integers, this corollary gives exactly the number of linearly independent rational functions $\sigma_1, \cdots, \sigma_\mu$ which have no singularities except possibly poles of orders $\geqq -\mu_k$ at the cycles P_k, provided that

$$q = \mu_1 + \cdots + \mu_s > 2p-2.$$

To prove this statement we have only to apply the Corollary 2 to the divisor $Q = P_1^{\mu_1}, \cdots, P_s^{\mu_s}$. When the sum q is less than or equal to $2p-2$, we can examine the divisors $Q' = Q_\xi X/D^2Q$. If the number μ' for one of them can be determined, then μ is again known from the first of equations (25).

THE UNIVERSITY OF CHICAGO.

AN EXTENSION OF THE DEFINITION OF THE GREEN'S FUNCTION IN ONE DIMENSION.

By W. M. Whyburn.

The Green's function in one dimension was defined by Bôcher in 1901.* This definition applies to linear differential systems of the nth order where the coefficients of the differential equation are continuous functions and the coefficients of the n linearly independent boundary conditions are constants.

It is the purpose of this paper to extend this definition to apply to linear differential systems of the second order whose coefficients are analytic functions of a parameter. This extended definition of the Green's function is used to carry over the results that have been obtained for these differential systems† to integral equations whose kernels are in general analytic functions of a parameter.‡

Consider the general self-adjoint linear homogeneous differential equation of the second order

$$\frac{d}{dx}\left[K(x,\lambda)\frac{du}{dx}\right] - G(x,\lambda)u = 0, \tag{1}$$

together with the self-adjoint boundary conditions

$$M_j[u(a,\lambda)] = M_j[u(b,\lambda)], \qquad (j = 1, 2), \tag{2}$$

where

$$M_j[u(x,\lambda)] = \alpha_j(x,\lambda)u(x,\lambda) + \beta_j(x,\lambda)K(x,\lambda)\frac{d}{dx}u(x,\lambda),$$

$$\alpha_1(a)\beta_2(a) - \alpha_2(a)\beta_1(a) = \alpha_1(b)\beta_2(b) - \alpha_2(b)\beta_1(b).$$

In addition to the above conditions we will impose the following which must be satisfied:

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* Bôcher, Green's functions in space of one dimension, Bull. Amer. Math. Soc., vol. 7 (1901), p. 297.

† Ettlinger, Existence theorems for the general real self-adjoint linear system of the second order. Trans. Amer. Math. Soc., vol. 19 (1918), pp. 79–96, vol. 22 (1921), pp. 136–143.

‡ This paper was written at the suggestion of and under the supervision of Professor H. J. Ettlinger.

I. $K(x, \lambda)$, $G(x, \lambda)$, $\alpha_j(x, \lambda)$, $\beta_j(x, \lambda)$, $\alpha'_{jx}(x, \lambda)$, $\beta'_{jx}(x, \lambda)$, are continuous real functions of x in the interval

$$(X) \qquad a \leqq x \leqq b$$

and for all real values of λ in the interval*

$$(L) \qquad L_1 < \lambda < L_2.$$

II. $K(x, \lambda)$ is positive everywhere in (X, L) and

$$|\alpha_j(x, \lambda)| + |\beta_j(x, \lambda)| > 0 \text{ in } (X, L).$$

III. For each value of x in (X) K and G decrease (at least do not increase) as λ increases.

IV. Either $\beta_j \equiv 0$ for all values of x and λ in (X, L); or else $\beta_j \not\equiv 0$ in $a < x < b$ for all λ's in (L) and one of the following is true,

either $(a)\beta_j(a) \equiv 0$ for all λ's in $(L)\beta_j(b) \not\equiv 0$, $\dfrac{\alpha_j(b)}{\beta_j(b)}$ decreases as λ increases;

or $(b)\beta_j(b) \equiv 0$, $\beta_j(a) \not\equiv 0$ for all λ in (L) and $\dfrac{\alpha_j(a)}{\beta_j(a)}$ decreases (or does not increase) as λ increases;

or $(c)\beta_j(a) \not\equiv 0$, $\beta_j(b) \not\equiv 0$, $\dfrac{\alpha_j(a)}{\beta_j(a)}$ and $\dfrac{\alpha_j(b)}{\beta_j(b)}$ decrease (or do not increase) as λ increases.

V.

$$\lim_{\lambda = L_1}\left[-\frac{\min G}{\min K}\right] = -\infty \quad \text{and} \quad \lim_{\lambda = L_2}\left[-\frac{\max G}{\max K}\right] = +\infty.$$

Let us write down the equation

$$(3) \qquad \frac{d}{dx}\left[K(x, \lambda)\frac{du}{dx}\right] - [G(x, \lambda) + g(x, \lambda)]u = 0,$$

which we will define as the reduced equation corresponding to equation (1). We define the differential system (3), (2) as the reduced system for the system (1), (2). $g(x, \lambda)$ is a positive function to be defined later. A value of λ, $\lambda = \bar{\lambda}$, for which the differential system has a solution other than

* In particular L_1 may be $-\infty$ and L_2 may be $+\infty$.

the identically vanishing solution is called a characteristic number of the system. There exists an infinite number of characteristic numbers for the system (1), (2).* We wish to obtain an expression for the solution of this system by means of the Green's function for the reduced system. In order to set up the Green's function for the reduced system, (3), (2), it is necessary† that this system be incompatible for the particular value of λ that we are considering. Hence we must define $g(x, \lambda)$ so that the reduced system will be incompatible for the values of λ for which the given system, (1), (2) is compatible, that is, for the characteristic numbers, l_i, of the system. Substitute the ith characteristic number into the system (1), (2)

$$(1') \qquad \frac{d}{dx}\left[K(x, l_i)\frac{d}{dx}u(x, l_i)\right] - G(x, l_i)\,u(x, l_i) = 0,$$

$$(2') \qquad M_j[u(a, l_i)] = M_j[u(b, l_i)] \qquad (j = 1, 2).$$

System (1'), (2') has a solution. Bôcher‡ has shown that it is possible to choose a continuous function of x, $g(x, l_i)$, such that

$$0 < g(x, l_i) < \epsilon_i,$$

where ϵ_i is an arbitrarily assigned positive constant, and such that the system

$$(3') \qquad \frac{d}{dx}\left[K(x, l_i)\frac{d}{dx} - u(x, l_i)\right] - [G(x, l_i) + g(x, l_i)]\,u(x, l_i) = 0,$$

$$(2') \qquad M_j[u(a, l_i)] = M_j[u(b, l_i)]$$

is incompatible. Not only can $g(x, l_i)$ be defined in this way but it can be chosen in an infinite number of ways for any given ϵ_i. We will choose $g(x, l_i)$ by the above method and in addition we will choose our ϵ_i so as to restrict it further.

Since $g(x, l_i)$ is a positive continuous function of x in the closed interval (X) it must have an absolute minimum in that interval which is greater than zero. Call this absolute minimum of $g(x, l_i)$, ϵ_{i+1}, and choose $g(x, l_{i+1})$ so that

$$0 < g(x, l_{i+1}) < \epsilon_{i+1}$$

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* Ettlinger, Oscillation theorems for the real self-adjoint linear system of the second order, Trans. Amer. Math. Soc., vol. 22 (1921), pp. 136-143.

† Bôcher, Leçons sur les Méthodes de Sturm, pp. 98-107.

‡ Bôcher, On small variations in the coefficients of differential systems, Bull. Amer. Math. Soc., vol. 20 (Oct. 1914), p. 1.

throughout (X) for $i = 1, 2, 3, \cdots$. We have now defined $g(x, l_i)$ so that

$$\epsilon_1 > g(x, l_1) > g(x, l_2) > g(x, l_3) > \cdots > g(x, l_i) > \cdots > 0,$$

and so that the system (3′), (2′) is incompatible for each characteristic number of the system (1), (2). Now define $g(x, \lambda)$ in such a way that it shall take on the value $g(x, l_i)$ at $\lambda = l_i$ for each l_i in (L) and so that $g(x, \lambda)$ shall be a positive continuous function of x and λ in (X, L).

$$(4) \quad g(x, \lambda) = g(x, l_i) + \frac{\lambda - l_i}{l_{i+1} - l_i}[g(x, l_{i+1}) - g(x, l_i)] \text{ in } l_i \leqq \lambda \leqq l_{i+1}.$$

Equation (4) defines $g(x, \lambda)$ completely* as a function of x and λ so that it has the following properties:

A. It is a continuous function of x and λ in (X, L).

B. It is a non-increasing function of λ.

C. It is positive throughout (X, L).

D. $g(x, \lambda)$ takes on the value $g(x, l_i)$ at $\lambda = l_i$ for each l_i in (L).

E. It makes the reduced system (3), (2) incompatible for those values of λ, $\lambda = l_i$, for which the system (1), (2) is compatible. Hence, when the given system is compatible, the reduced system is incompatible.

Since the system (2), (3) is incompatible when the system (1), (2) is compatible we will take a point ξ in the interval (X) and try to determine a function, u, which satisfies the boundary conditions (2), is continuous throughout (X), and has a first derivative that is continuous everywhere in the intervals $a \leqq x < \xi$ and $\xi < x \leqq b$ but such that at $x = \xi$ it has a finite jump that is given by

$$u'_x(\xi + 0, \lambda) - u'_x(\xi - 0, \lambda) = \frac{1}{K(\xi, \lambda)}$$

and finally satisfies equation (3) at every point of (X) except at the point $x = \xi$. We will call this function $\Gamma(x, \xi; \lambda)$. This function is uniquely determined by the method used by Bocher.† $\Gamma(x, \xi; \lambda)$ exists and is continuous provided the determinant

$$\begin{vmatrix} M_1[u_1(a, \lambda)] - M_1[u_1(b, \lambda)] & M_1[u_2(a, \lambda)] - M_1[u_2(b, \lambda)] \\ M_2[u_1(a, \lambda)] - M_2[u_1(b, \lambda)] & M_2[u_2(a, \lambda)] - M_2[u_2(b, \lambda)] \end{vmatrix} \neq 0.$$

* Throughout the interval $L_1 < \lambda < l_1$ we define $g(x, \lambda)$ to have the same value as at $\lambda = l_1$.

† Bôcher, loc. cit., pp. 98–102.

This determinant is zero only for those values of λ which are the characteristic numbers of the reduced system, that is, when (2), (3) is compatible. But when the given system (1), (2) is compatible the reduced system (2), (3) is incompatible, hence $\Gamma(x, \xi; \lambda)$ exists and is continuous for all of the characteristic numbers of the system (1), (2).

It can easily be shown that the coefficients of the reduced system (3), (2) satisfy all of the conditions I–V and hence there exists an infinite set of characteristic numbers, $\bar{l}_i$, for which values of λ this system is compatible. For these values of λ, $\lambda = \bar{l}_i$, the above determinant is zero and $\Gamma(x, \xi; \lambda)$ does not exist but has a pole.*

$\Gamma(x, \xi; \lambda)$ is called the Green's function for the system and has the following properties:

A. It is a continuous function of x throughout (X).

B. It has a first derivative that is continuous in $a \leqq x < \xi$ and $\xi < x \leqq b$ but which has a jump of $1/K(\xi, \lambda)$ at $x = \xi$.

C. It satisfies the boundary conditions (2).

D. It satisfies the differential equation (3) at every point of (X) except at the point $x = \xi$.

E. It exists and is continuous for every value of λ in (L) except at $\lambda = \bar{l}_i$, where it has a pole.

We can now write down an integral equation whose solution is the solution of the given differential system

$$u(x, \lambda) = -\int_a^b \Gamma(x, \xi; \lambda) g(\xi, \lambda) u(\xi, \lambda)\, d\xi. \tag{5}$$

$u(x, \lambda)$ is the solution of the system (1), (2) as may be shown by substituting its value as given by equation (5) in the differential system (1), (2). Hence every solution of the integral equation is a solution of the given differential system. It may also be shown† that every solution of the given differential system is a solution of the integral equation. Hence the differential system (1), (2) and the integral equation (5) are equivalent, so that we may state the same existence theorem for the integral equation (5) that has been proved‡ for the system (1), (2). The integral equation has the same characteristic numbers as the differential system. Equation (5) may be transformed into an equation having a symmetric kernel by the transformation

$$v(x, \lambda) = u(x, \lambda) \sqrt{g(x, \lambda)},$$

* Birkhoff, Boundary value and expansion problems of ordinary linear differential Equations, Trans. Amer. Math. Soc., vol. 9 (1908), p. 377.

† See Bôcher, loc. cit., pp. 107–108.

‡ See first footnote, page 127.

since $g(x, \lambda) > 0$, and the resulting integral equation has the same characteristic numbers as the given differential system.

If we begin with the non-homogeneous differential system

$$\frac{d}{dx}\left[K(x, \lambda)\frac{du}{dx}\right] - G(x, \lambda)u = r(x, \lambda), \tag{6}$$

$$M_j[u(a, \lambda)] - M_j[u(b, \lambda)] = \gamma_i(\lambda), \qquad (j = 1, 2) \tag{7}$$

where $r(x, \lambda)$ is a continuous function of x and λ in (X, L), $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$ are continuous functions of λ in (L), and conditions I–V are satisfied, we arrive at an integral equation of the type

$$v(x, \lambda) = f(x, \lambda) + \int_a^b H(x, \xi; \lambda)v(x, \lambda)\, d\xi, \tag{8}$$

which is an integral equation of the second kind whose kernel is a continuous function of λ except for poles at $\lambda = l_i$. This integral equation is equivalent to system (6), (7) and has solutions for all values of λ except for $\lambda = l_i$, the characteristic numbers of the system (1), (2) and the integral equation (5).

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REAL REPRESENTATIONS OF ANALYTIC COMPLEX CURVES.*

By W. C. Graustein.

In a previous paper† the author determined all the representations of a complex point by pairs of ordered real points which satisfy certain general conditions. In the present paper application is made to analytic complex curves, in the plane and in three-dimensional space, of certain of these representations, namely, those in which the coördinates of the two real points are linear in the coördinates of the complex point.

In the plane let (x, y), where $x = x_1 + ix_2$ and $y = y_1 + iy_2$, be the complex point and let $(X, Y) \rightarrow (X', Y')$ be the ordered pair of real points representing it. Then the representations with which we are concerned are defined by the equations

$$(1) \qquad \begin{aligned} X &= x_1 + kx_2 + ly_2, & X' &= x_1 - kx_2 - ly_2, \\ Y &= y_1 - lx_2 + ky_2, & Y' &= y_1 + lx_2 - ky_2, \end{aligned}$$

where k and l are real parameters, not both zero.

Of these representations there are two important subsets: those for which $l = 0$, and those for which $k = 0$. The former contain the two representations of Marie ($l = 0$, $k^2 = 1$) and shall be called, for convenience, *the generalized Marie representations.* The latter, of which the representations of Laguerre ($k = 0$, $l^2 = 1$) are special cases, shall be known as *the generalized Laguerre representations.*‡

Since an arbitrary point of a complex curve depends on two real parameters, u and v, a representation (1) of the curve will be, in general, a real transformation of the real plane, defined parametrically: X, Y and X', Y' will be given as functions of u and v. The facts, necessary for our purposes, concerning transformations thus defined are summarized in § 1.

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* Presented to the American Mathematical Society, September 7, 1922.

† Representations of a complex point by pairs of ordered real points, Trans. Amer. Math. Soc., vol. 24 (1922), pp. 245–254.

‡ Characteristic of the representations (1) is that they are invariant under the group of real direct transformations of similarity. The generalized Marie representations are also invariant under the real indirect transformations of similarity, whereas the generalized Laguerre representations are inverted by these transformations,—a representation $(X, Y) \rightarrow (X', Y')$ is said to be inverted by a transformation if it is carried by it into the representation $(X', Y') \rightarrow (X, Y)$. Cf. Author, loc. cit.

Inasmuch as (1) can always be solved uniquely for x and y, every transformation of the plane will represent ∞^2 complex points, that is, a membrane* of points. In order to determine when this membrane of points is, in particular, a curve, it is necessary to have at hand certain theorems concerning membranes (§ 2).

The developments in §§ 1, 2 render the discussion of the representations (1) of plane analytic complex curves systematic and transparent. In § 3, questions of a general nature are treated and certain types of representations, e. g., the generalized and special representations of Marie and Laguerre, are characterized,† whereas in § 4 the particular question of the representations of straight lines is studied.

A representation of a complex point in space by an ordered pair of real points, when applied to a complex curve, leads in general to a map of one real surface upon another. In § 5 it is shown that this map, in the case of the representations in which we are interested, is precisely that which exists between two associate translation surfaces with conjugate-imaginary generators.

1. **Concerning real transformations of the plane, parametrically defined.**‡ Let the coördinates (X, Y) and (X', Y') of the given and transformed points of a real analytic transformation of the plane be (real, analytic) functions of the two real parameters, u, v. The four functions, taken in pairs, yield six Jacobians, e. g.,

$$(XY) = X_u Y_v - X_v Y_u,$$

and these are connected by the relation

$$(XY)(X'Y') = (XX')(YY') - (XY')(YX').$$

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* A manifold of ∞^2 complex points depending analytically on two real parameters has been given the name *tela* (Italian) by Segre; *Membran* (German) by Study; and *congruence* by Coolidge. Cf. E. Study, Vorlesungen über ausgewählte Gegenstände der Geometrie, erstes Heft: Ebene analytische Kurven und zu ihnen gehörige Abbildungen, p. 24; J. L. Coolidge, Differential Geometry of the complex plane, Trans. Amer. Math. Soc., vol. 23 (1922), p. 118.

† The special representations of Marie and Laguerre have been exaustively treated by Study, loc. cit. It is not our purpose to duplicate Study's discussion, with its wealth of detail, for the other representations (1), but merely to develop the results of first importance. The methods used differ from those of Study in the emphasis placed on the idea of a curve as a special case of a membrane, and in the application of the theory of parallel maps to transformations of the plane (§ 2).

‡ Cf. Study, loc. cit., pp. 53–61. For orientation concerning Theorems 2–5, in which the transformation is considered as a parallel map, cf. Author, Parallel maps of surfaces, Trans. Amer. Math. Soc., vol. 23 (1922), pp. 299–301.

From the Jacobians form the expressions:

$$(2)\quad \begin{aligned} A_1 &= (XY)+(X'Y'), \quad M_1 = (XX')+(YY'), \quad M_2 = (XY')-(YX'); \\ A_2 &= (XY)-(X'Y'), \quad N_1 = (XX')-(YY'), \quad N_2 = (XY')+(YX'), \end{aligned}$$

for which

$$(3)\qquad A_1^2+N_1^2+N_2^2 = A_2^2+M_1^2+M_2^2.$$

The Jacobians, and hence the expressions (2), are relative invariants of a change of parameters. Moreover, A_1, A_2, M_1, M_2, $N_1^2+N_2^2$ are absolute invariants with respect to the group of real motions. In terms of them it is possible to express all the important invariantive properties of the transformation.

THEOREM I. *The transformation is singular if and only if* $A_1^2-A_2^2=0$.

Assume henceforth that the transformation is non-singular.

THEOREM 2. *The transformation is direct or indirect according as* $A_1^2-A_2^2>0$ or <0.

If corresponding curves always have parallel tangents at corresponding points, the transformation is homothetic. In a non-homothetic transformation there exist two corresponding systems of curves such that corresponding curves have parallel tangents at corresponding points. According as the two families of curves which constitute each system are real and distinct, real and coincident, or conjugate-imaginary, the transformation shall be termed hyperbolic, parabolic, or elliptic.

THEOREM 3. *The transformation is homothetic if and only if* $M_1 = 0$ *and* $M_2^2-A_1^2+A_2^2 = 0$. *A non-homothetic transformation is hyperbolic, parabolic, or elliptic, according as* $M_2^2-A_1^2+A_2^2>0$, $=0$, or <0.

THEOREM 4. *The systems of corresponding parallel curves of a non-homothetic transformation are orthogonal if and only if* $M_1 = 0$.

The cross ratio in which two corresponding directions of departure from two corresponding points of a non-homothetic transformation divide the corresponding points of a non-homothetic transformation divide the corresponding parallel directions at these points is known as the invariant of the transformation. The value of the invariant is

$$(4)\qquad I = \frac{M_2-\sqrt{M_2^2-A_1^2+A_2^2}}{M_2+\sqrt{M_2^2-A_1^2+A_2^2}}.$$

THEOREM 5. *The invariant of a transformation is* -1 *if and only if* $M_2 = 0$.

THEOREM 6. *Necessary and sufficient that the transformation be*
(a) *directly (indirectly) equi-areal is that* $A_2 = 0$ $(A_1 = 0)$;
(b) *directly (indirectly) conformal is that* $N_1 = N_2 = 0$ $(M_1 = M_2 = 0)$.

2. Concerning complex plane curves as special cases of membranes of points. The equations

$$x = x(u, v), \qquad y = y(u, v), \tag{5}$$

where $x(u, v), y(u, v)$ are complex analytic functions of the real parameters u, v, represent in general a membrane of points. From the functions $x(u, v), y(u, v)$ and their conjugates, $\bar{x}(u, v), \bar{y}(u, v)$, six Jacobians can be formed, and then combined to give:

$$\begin{aligned} &\lambda_1 = (xy) + (\bar{x}\bar{y}), \quad i\mu_1 = (x\bar{x}) + (y\bar{y}), \quad \mu_2 = (x\bar{y}) - (y\bar{x}), \\ &i\lambda_2 = (xy) - (\bar{x}\bar{y}), \quad i\nu_1 = (x\bar{x}) - (y\bar{y}), \quad i\nu_2 = (x\bar{y}) + (y\bar{x}). \end{aligned} \tag{6}$$

The quantities $\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2$ are all real and satisfy the identity

$$\mu_1^2 + \lambda_1^2 + \lambda_2^2 = \mu_2^2 + \nu_1^2 + \nu_2^2; \tag{7}$$

the first four, together with $\nu_1^2 + \nu_2^2$, are absolute invariants of the group of real motions.

THEOREM 7. *Equations* (5) *represent less than* ∞^2 *points if and only if* $\lambda_1 = \lambda_2 = \mu_1 = 0$.

THEOREM 8. *Equations* (5) *represent a curve, which is then necessarily analytic, if and only if* $\lambda_1 = \lambda_2 = 0$, $\mu_1 \neq 0$.

THEOREM 9. *A curve represented by* (5) *is a straight line if and only if* μ_1, μ_2 *satisfy a linear homogeneous equation with constant coefficients:*

$$c_1\mu_1 + c_2\mu_2 = 0.$$

Theorems 7, 8 are novel only in form.* Theorem 9 is believed to be original and is worthy of exposition.

For the line $x = \text{const.}$, $\mu_2 = 0$; for $y = ax + b$,

$$i(a - \bar{a})\mu_1 + (1 + a\bar{a})\mu_2 = 0.$$

* For a proof of Theorem 8, cf., e. g., Study, loc. cit., p. 43.

Conversely, let $y = \varphi(x)$ be an analytic curve for which $\mu_2/\mu_1 = c$. Equations (5) then become

$$x = x_1 + ix_2, \qquad y = \varphi_1(x_1, x_2) + i\varphi_2(x_1, x_2),$$

where, since $\varphi(x)$ is analytic,

$$\frac{\partial \varphi_1}{\partial x_1} = \frac{\partial \varphi_2}{\partial x_2}, \qquad \frac{\partial \varphi_1}{\partial x_2} = -\frac{\partial \varphi_2}{\partial x_1}, \tag{8}$$

and

$$\frac{\partial^2 \varphi_2}{\partial x_1^2} + \frac{\partial^2 \varphi_2}{\partial x_2^2} = 0. \tag{9}$$

By virtue of (8) the condition $\mu_2/\mu_1 = c$ can be written:

$$\left(\frac{\partial \varphi_2}{\partial x_1}\right)^2 + \left(\frac{\partial \varphi_2}{\partial x_2}\right)^2 + 1 = \frac{2}{c}\frac{\partial \varphi_2}{\partial x_1}. \tag{10}$$

The problem, then, consists in solving (9) and (10) simultaneously. The equation obtained by eliminating $\partial^2 \varphi_2/\partial x_2^2$ from (9) and the equation resulting from (10) by partial differentiation with respect to x_2 can be integrated with respect to x_1, giving

$$\frac{\partial \varphi_2}{\partial x_1} - \frac{1}{c} = f(x_2)\frac{\partial \varphi_2}{\partial x_2}. \tag{11}$$

From equations (9), (10), (11) it is readily proved that $\partial \varphi_2/\partial x_1$ and $\partial \varphi_2/\partial x_2$ are constant; if $c = 0$, the same result follows from (9) and (10) alone. Hence, in any case, $d\varphi/dx$ is constant and φ is a linear function of x.

COROLLARY. *If the slope of the line is infinite,* $\mu_2 = 0$; *if its slope is* a,

$$i(a - \bar{a})\mu_1 + (1 + a\bar{a})\mu_2 = 0.$$

From this result follow immediately the theorems:

THEOREM 10. *A curve represented by* (5) *is a line with real direction if and only if* $\mu_2 = 0$.

THEOREM 11. *A curve represented by* (5) *is a right-handed (left-handed)* minimal line if and only if* $\mu_1 - \mu_2 = 0$, $(\mu_1 + \mu_2 = 0)$.

3. **Representations of plane curves.** The representation by means of (1) of a curve or a membrane defined by the equations,

* The minimal lines, $x + iy + c = 0$ and $ix + y + c = 0$, are right-handed and left-handed, respectively.

(5) $$x = x_1(u,v) + ix_2(u,v), \qquad y = y_1(u,v) + iy_2(u,v),$$

is the transformation $(X, Y) \to (X', Y')$ of the plane given by

(12) $$\begin{aligned} X &= x_1 + kx_2 + ly_2, & X' &= x_1 - kx_2 - ly_2, \\ Y &= y_1 - lx_2 + ky_2, & Y' &= y_1 + lx_2 - ky_2, \end{aligned} \qquad k^2 + l^2 \neq 0,$$

where x_1, x_2, y_1, y_2 have the values accorded them by (5).

If (5) represents a curve (Theorem 8), then

$$4(XY) = (1 + k^2 + l^2)\mu_2 + 2l\mu_1, \qquad 4(X'Y') = (1 + k^2 + l^2)\mu_2 - 2l\mu_1.$$

The transformation (12) is, therefore, singular if and only if

$$2l\mu_1 \pm (1 + k^2 + l^2)\mu_2 = 0,$$

i. e., by Theorem 9, Corollary, if and only if the curve is a straight line whose slope a satisfies one of the equations:

(13) $$\frac{i(a - \bar{a})}{1 + a\bar{a}} = \pm \frac{i(\omega - \bar{\omega})}{1 + \omega\bar{\omega}}, \qquad \omega = k + il,$$

or one of the equivalent equations:

$$\frac{(a \mp i)(\bar{a} \pm i)}{(a \pm i)(\bar{a} \mp i)} = \frac{(\omega - i)(\bar{\omega} + i)}{(\omega + i)(\bar{\omega} - i)}.$$

THEOREM 12. *No representation, $\omega = k + il$, of a curve which is not a straight line is singular. But every representation degenerates for certain straight lines: a Laguerre representation, $\omega = \pm i$, for lines of two slopes, the isotropics; every other representation for lines of infinitely many slopes, namely the slopes a such that a and $\bar{a}$ form with i and $-i$ the same cross ratio as do ω and $\bar{\omega}$.*

Both equations (13) are satisfied when and only when $\omega = \bar{\omega}$ and $a = \bar{a}$. In other words, both point-fields, (X, Y) and (X', Y'), contain less than ∞^2 points if and only if the representation is a generalized Marie representation and the line is a line with real direction.*

* In this case each point-field consists of the points on a line; the lines are coincident or parallel, according as the given line is real or imaginary, and to each point of one corresponds every point of the other. In a Laguerre representation of a minimal line one point-field consists merely of the real point of the line, to which corresponds every point in plane. In every other degenerate representation of a line one point-field consists of all the points of a line, whereas the other is unrestricted.

Henceforth it will be tacitly assumed that the representations considered are non-degenerate. To apply the theory of §§ 1, 2, it is necessary first to evaluate A_i, M_i, N_i for (12) in terms of λ_i, μ_i, ν_i for (5), and vice versa:

$$
\begin{aligned}
2A_1 &= \lambda_1 + \mu_2 - r^2(\lambda_1 - \mu_2), & 2r^2\lambda_1 &= r^2(A_1 + M_2) - (A_1 - M_2),\\
2M_2 &= \lambda_1 + \mu_2 + r^2(\lambda_1 - \mu_2), & 2r^2\mu_2 &= r^2(A_1 + M_2) + (A_1 - M_2),\\
M_1 &= k\mu_1 - l\lambda_2, & r^2\mu_1 &= kM_1 + lA_2,\\
A_2 &= l\mu_1 + k\lambda_2, & r^2\lambda_2 &= -lM_1 + kA_2,\\
N_1 &= k\nu_1 + l\nu_2, & r^2\nu_1 &= kN_1 - lN_2,\\
N_2 &= -l\nu_1 + k\nu_2, & r^2\nu_2 &= lN_1 + kN_2,
\end{aligned}
\tag{14}
$$

where

$$r^2 = k^2 + l^2.$$

In these equations set $\lambda_1 = \lambda_2 = 0$, $\mu_1 \neq 0$, i. e., assume that (5) represents a curve.

Since $M_1 = k\mu_1$, $M_2^2 - A_1^2 + A_2^2 = l^2\mu_1^2 - r^2\mu_2^2$, the conditions in Theorem 3 that (12) be homothetic reduce to $k = 0$, $\mu_1^2 - \mu_2^2 = 0$.

THEOREM 13. *A curve is represented by a homothetic transformation if and only if it is minimal line and the representation is a generalized Laguerre representation.*

If $l = 0$, $A_2 = 0$ and conversely. Consequently, by Theorem 6:

THEOREM 14. *A generalized Marie representation of a curve is always directly equi-areal and is the only type of representation of a curve ever having this property.*

Furthermore, if $l = 0$, $A_2 = 0$ and M_2/A_1 is a constant less in absolute value than unity. Therefore, by Theorem 3 and formula (4):

COROLLARY. *A generalized Marie representation of a curve is always elliptic and of constant invariant.*

If $k = 0$, $M_1 = 0$ and conversely. Thus, by Theorem 4:

THEOREM 15. *A generalized Laguerre representation of a curve always possesses orthogonal systems of corresponding parallel curves and is the only type of representation of a curve ever having this property.**

The transformation (12) has the invariant -1 (Theorem 5) if and only if $\mu_2 = 0$ or $r^2 = 1$.

THEOREM 16. *Characteristic of a representation for which $k^2 + l^2 = 1$ is that, when applied to curves, its invariant is always -1, regardless of the curve taken.*

* A generalized Laguerre representation is always hyperbolic, except in the case of a minimal line when it is homothetic; cf. Theorem 13.

Combining Theorem 16 with Theorems 14 and 15 in turn gives the following results:

THEOREM 17. *Characteristic of a (special) Marie representation of a curve is that it is always directly equi-areal and has the invariant* -1.

THEOREM 18. *Characteristic of a (special) Laguerre representation of a curve is that it is always inversely conformal.*

Theorems 14—18 give the characteristic properties of various types of representations of curves with respect to the totality of representations (12) of curves. This totality, however, is not that of all non-singular transformations of the plane. It is true that an arbitrary non-singular transformation $(X, Y) \rightarrow (X', Y')$ will always represent a unique membrane by any given representation, inasmuch as equations (12), for any given values of k and l (not both 0), can always be solved uniquely for x_1, x_2, y_1, y_2. *But this membrane is a curve if and only if* $\lambda_1 = \lambda_2 = 0$,* *i. e., if and only if*

$$r^2(A_1 + M_2) - (A_1 - M_2) = 0, \qquad lM_1 - kA_2 = 0, \tag{15}$$

If $l = 0$ $(k \neq 0)$, these equations reduce to

$$A_2 = 0, \qquad M_2/A_1 = c, \qquad c^2 < 1. \tag{16}$$

Conversely, given (16), equations (15) are satisfied, when $l = 0$, if and only if

$$k^2 = \frac{1-c}{1+c}.$$

But (16), according to the proofs of Theorem 14 and its corollary, are sufficient conditions that the given transformation be directly equi-areal, elliptic, with constant invariant; they are also necessary, as can be readily shown. In particular, if the invariant is -1, then $M_2 = 0$ or $c = 0$, whence $k^2 = 1$, and conversely.

THEOREM 19. *A non-singular transformation of the plane is a generalized Marie representation of a curve if and only if it is directly equi-areal, elliptic, with constant invariant. It is a (special) Marie representation of a curve if and only if it is directly equi-areal with invariant* -1. *In each case two representations, inverses of one another, are determined and, corresponding to them, two curves, conjugate to each other.*

Equations (15) reduce, for $k = 0$ and $l^2 = 1$, to $M_1 = M_2 = 0$. Thus, by Theorem 6,

* Since the given transformation is assumed non-singular, it is unnecessary to demand also (Theorem 8) that $\mu_1 \neq 0$.

THEOREM 20. *A non-singular transformation of the plane is a (special) Laguerre representation of a curve if and only if it is inversely conformal.*

4. Representations of straight lines. If x, y in (5) are linear in u and v, so also are X, Y, X', Y' in (12), and conversely. Hence:

THEOREM 21. *A straight line is always represented by an affine transformation and is the only type of curve ever thus represented.*

Restricting the discussion to the representations of curves, set $\lambda_1 = \lambda_2 = 0$, $\mu_1 \neq 0$ in (14).

If $\mu_2 = 0$, $A_1 = 0$, and conversely; also, if $\mu_2 = 0$, $M_2 = 0$. Hence, by Theorems 5, 6, 10:

THEOREM 22. *A straight line with real direction is always represented by an indirectly equi-areal transformation of invariant -1. It is the only type of curve ever represented by an indirectly equi-areal transformation.*

If $\mu_1^2 - \mu_2^2 = 0$ (i. e., if $\nu_1^2 + \nu_2^2 = 0$), $N_1 = N_2 = 0$, and conversely. Thus, by Theorems 6, 11:

THEOREM 23. *A minimal line is always represented by a directly conformal transformation and is the only type of curve ever thus represented.*

Combining Theorems 14 and 23, and Theorems 15 and 22, we get:

THEOREM 24. *A generalized Marie representation of a minimal line is directly isometric, and a generalized Laguerre representation of a line with real direction is indirectly isometric. It is only in these cases that a representation of a curve is ever isometric.*

According to Theorems 13, 24, a translation, being at once homothetic and directly isometric, can never serve as a representation of a curve. Hence not all affine transformations of the plane are representations of straight lines. In this connection, however, certain interesting results can be obtained.

Minimal lines. According to Theorems 21, 23, a minimal line is represented by a direct transformation of similarity, which, by Theorems 13, 24, cannot be involutory or a translation. Conversely, given an arbitrary direct transformation of similarity, not at once equi-areal and homothetic. Then A_i, M_i, N_i can be considered constant; moreover, $A_2^2 + M_1^2 \neq 0$ and $A_1^2 - M_2^2 > 0$. Under these conditions equations (15) have always two distinct solutions for k, l, the constants in one being the negatives of those in the other. That is, the given transformation represents a curve by each of two inverse representations. The two curves are, by Theorem 23, minimal lines, and these minimal lines are necessarily those through the fixed point of the given transformation.

THEOREM 25. *The representation of a minimal line is a direct transformation of similarity, not involutory or a translation. Conversely, every such transformation represents a unique minimal line of each type by a unique*

representation; *the two minimal lines are conjugate and the two representations inverse.*

Consider a given minimal line and the ∞^2 direct transformations of similarity of the type in question having the real point of the minimal line as fixed point. Each of these transformations represents the minimal line by a unique representation, and conversely.

THEOREM 26. *There is a one-to-one correspondence between the* ∞^2 *representations of a given minimal line and the* ∞^2 *direct transformations of similarity*:

$$Z' - z_0 = \alpha'(Z - z_0), \qquad (\alpha'^2 \neq 1), \tag{17}$$

where z_0 *is the real point of the minimal line.*

Here, the Gauss plane has been introduced by setting $Z = X + iY$, $Z' = X' + iY'$. Then (12) can be written:

$$Z = x_1 + iy_1 - i\alpha(x_2 + iy_2), \qquad Z' = x_1 + iy_1 + i\alpha(x_2 + iy_2), \tag{18}$$

where

$$\alpha = l + ik.$$

For the right-handed and left-handed minimal lines through the origin,

$$x_2 + iy_2 = i(x_1 + iy_1) \text{ and } x_2 + iy_2 = -i(x_1 + iy_1),$$

whence, from (18),

$$Z' = \frac{1-\alpha}{1+\alpha} Z \text{ and } Z' = \frac{1+\alpha}{1-\alpha} Z,$$

provided $\alpha^2 \neq 1$, i. e., provided the representation is not one of Laguerre (Theorem 12).

COROLLARY. *The one-to-one correspondence between the representations* (18), $\alpha^2 \neq 1$, *and the transformations* (17) *is defined by*

$$\alpha' = \frac{1-\alpha}{1+\alpha} \quad \text{or} \quad \alpha' = \frac{1+\alpha}{1-\alpha}, \tag{19}$$

according as the minimal line is right-handed or left-handed.

By Theorem 24, a generalized Marie representation of a minimal line is a non-involutory rotation, and conversely. If ψ is defined by $\tan \psi = k$, where $\pi/2 > |\psi| > 0$, the angle of rotation is equal, by (19), to -2ψ or 2ψ, according as the given minimal line is right-handed or left-handed.

A generalized Laguerre representation of a minimal line (Theorem 13) is a stretching with algebraic coefficient m, $\neq \pm 1$, and conversely. The value of m is $(1-l)/(1+l)$ or $(1+l)/(1-l)$, according to the type of the line.

Lines with real directions. In this case (Theorems 21, 22) the representation is an indirectly equi-areal affine transformation with invariant -1. Conversely, given such a transformation, A_i, M_i, N_i can be considered constant, and $A_1 = M_2 = 0$, $A_2 \neq 0$. Equations (15) reduce to $k/l = M_1/A_2$. Thus the given transformation represents curves by ∞^1 representations; by Theorem 22, these curves are lines with real directions.

THEOREM 27. *The representation of a line with real direction is an indirectly equi-areal affine transformation with invariant* -1. *Conversely, every such transformation represents a line with real direction by each of* ∞^1 *representations.*

Since an indirect affine transformation is hyperbolic, there exist two distinct pencils, S_1, S_2, of real parallel lines, the members of each of which are permuted by the transformation. In one pencil, say S_1, the given and the transformed lines trace the pencil in opposite directions, whereas in S_2 they move in the same direction. Hence, in S_1 there must be a fixed (finite) line L, and corresponding points on L trace L in the same direction. If L is taken as the axis of x, the equations of the transformation are, then, of the form:

$$X' = aX + bY + c, \qquad Y' = eY, \qquad a > 0.$$

Necessary and sufficient that this transformation be indirectly equi-areal with invariant -1 is that $ae = -1$, $a + e = 0$; it thus becomes:

$$X' = X + bY + c, \qquad Y' = -Y. \tag{20}$$

An indirectly equi-areal affine transformation of invariant -1 *is the product of a simple shear along a line* L (*of coefficient* b), *a translation in the direction of* L (*of amount* c), *and a reflection in* L. The line L shall be called the *central line* of the transformation.

Comparison of (20) with the representation of the line $y = id$, namely:

$$X' = X + \frac{2k}{l}Y - 2\frac{k^2+l^2}{l}d, \qquad Y' = -Y,$$

leads to the following results.

THEOREM 28. *Let* l *be a line with real direction,* $\bar{l}$ *its conjugate, and* L *the real line midway between* l *and* $\bar{l}$. *The representation of* l *is an in-*

directly equi-areal affine transformation of invariant -1 *with* L *as central line. Given, conversely, such a transformation: if the translation is lacking* ($c = 0$), *the transformation represents* L *by each of the* ∞^1 *representations for which* $k/l = b/2$; *if the translation is present, the transformation represents any chosen line* l *(at a pure imaginary distance from* L*) by a unique representation and there exists a one-to-one correspondence between the* ∞^1 *lines* l *and the* ∞^1 *representations for which* $k/l = b/2$.

According to Theorem 24, a generalized Laguerre representation of l consists of a reflection in L, combined or not with a translation in the direction of L, according as l is imaginary or real; the converse is true.

5. **Generalized Marie representations of space curves.** The generalized Marie representation $X \to X'$ of the point x, with the coördinates x_1, x_2, x_3, can be written in the form:*

$$(21)\quad X_j = \frac{1-ik}{2}x_j + \frac{1+ik}{2}\bar{x}_j, \qquad X_j' = \frac{1+ik}{2}x_j + \frac{1-ik}{2}\bar{x}_j, \qquad (j = 1, 2, 3).$$

Let x trace an analytic curve,

$$x_j = x_j(u), \qquad (j = 1, 2, 3).$$

In general, X and X' will then trace surfaces, and the representation will be a map of the one surface upon the other. The surfaces degenerate into curves simultaneously, and then only when the directions dx_j/du, and $d\bar{x}_j/du$, $j = 1, 2, 3$, are always identical, i. e., only when the curve is a line with real direction.

Assuming that the curve is not a line with real direction, set:

$$(22)\qquad \frac{1-ik}{2}x_j = U_j(u), \qquad \frac{1+ik}{2}\bar{x}_j = \bar{U}_j(\bar{u}), \qquad (j = 1, 2, 3).$$

Then

$$(23)\quad X_j = U_j(u) + \bar{U}_j(\bar{u}), \qquad X_j' = e^{\theta i}U_j(u) + e^{-\theta i}\bar{U}_j(\bar{u}), \qquad (j = 1, 2, 3)$$

where

$$\tan\frac{\theta}{2} = k, \qquad 0 < |\theta| < \pi.$$

* The special Marie representations ($k^2 = 1$) of space curves was the subject of the author's thesis for the doctorate: Eine reelle Abbildung analytischer komplexer Raumkurven, Bonner Diss., Leipzig, 1913. The methods here used, to obtain the essential results for this and the generalized case, are much more powerful.

Thus, the surfaces X and X' are associate (non-symmetric) translation surfaces with conjugate-imaginary generators.*

Conversely, any two such surfaces form the representation of two conjugate-imaginary curves by two inverse representations, respectively. For, given θ and $U_j(u)$ in (23), then k from $\tan\theta/2 = k$ and x_j from (45) can be determined, either as

$$k_1 = \tan\frac{\theta}{2}, \quad x_j = \frac{2}{1-ik_1}U_j(u) \text{ or as } k_2 = -\tan\frac{\theta}{2}, \quad x_j = \frac{2}{1-ik_2}\bar{U}_j(\bar{u}).$$

THEOREM 29. *A necessary and sufficient condition that $X \to X'$ be a generalized Marie representation of a curve, not a line with real direction, is that X and X' trace two associate (non-symmetric) translation surfaces with conjugate-imaginary generators.*

Since $k = \pm 1$ if and only if $\theta = \pm\pi/2$:

Corollary. If the special Marie representations alone are considered, X and X' are two adjoint translation surfaces with conjugate-imaginary generators, and conversely.

From (22) it is clear that the generators are minimal curves if and only if the given curve is minimal.

THEOREM 30. *A necessary and sufficient condition that $X \to X'$ be a generalized Marie representation of a minimal curve is that X and X' be two associate (non-symmetric) minimal surfaces; or two adjoint minimal surfaces, if only the special representations of Marie are admitted.*

Two associate translation surfaces constitute a parallel map which can be completely characterized without reference to the nature of the surfaces.† This fact permits a restatement of Theorems 29, 30 in an entirely different form.

THEOREM 31. *Necessary and sufficient that $X \to X'$ be a generalized (special) Marie representation of a curve, not a line with real direction, is that X and X' be surfaces which correspond by an elliptic parallel map which is equi-areal and of constant invariant (of invariant -1). The curve is minimal, if and only if the map is also conformal.*

* Cf. Author, Parallel maps of surfaces, loc. cit., p. 313.

† Cf. Author, paper just cited, p. 313.

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NOTE
ON DIRICHLET SERIES WITH COMPLEX EXPONENTS.

By J. F. Ritt.

In a paper published some years ago*, I dealt with Dirichlet series $\sum a_n e^{\lambda_n z}$ in which the λ's are any complex numbers such that $\sum 1/|\lambda_n|$ is convergent. I proved that any analytic function which can be represented in some area in its domain of existence by a uniformly convergent series of this type is a uniform function, and that if it is analytic along the entire circumference of any circle, it is analytic also throughout the interior of that circle.

Hille†, using methods not resembling those of my paper, has recently shown that the circle in the above statement can be replaced by any Jordan curve, and that the domain of existence of the function is therefore simply connected.

I wish to point out here how Hille's result can be proved on the basis of a theorem given in § 9 of my paper. It is shown there that every function of the type described above admits of development into a series of linear combinations of exponentials which converges uniformly to the function in every finite area interior to the domain of existence of the function. This series converges uniformly, a fortiori, along any Jordan curve which lies in the domain of existence. As the linear combinations of exponentials are analytic throughout the interior of the Jordan curve, it follows from a well known theorem of Weierstrass that the series converges uniformly to an analytic sum over the interior of the curve.

* On a general class of linear homogeneous differential equations of infinite order with constant coefficients. Transactions of the American Mathematical Society, vol. 18 (1917), p. 21.

† Dirichlet series with complex exponents, Annals of Math., vol. 26 (1924), p. 261-278.

ON THE ORDER OF AN ANALYTIC FUNCTION AT A SINGULAR POINT.*

By M. H. Stone.

If there is given a Taylor's series converging, as we shall suppose, in the unit circle, it defines an analytic function

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots, \qquad z = re^{i\theta}, \qquad r < 1.$$

The study of the singularities of $f(z)$ on the circumference of the circle is clearly of the greatest importance. In his Thèse,† which is devoted to such a study, Hadamard associates with each point on the unit circle a quantity which he calls the order of $f(z)$ at that point. The researches of Hadamard and others‡ have shown that the behavior of $f(z)$ in the neighborhood of a point on the circumference of the circle of convergence is very closely connected with its order at that point. It is the purpose of the present paper to give a proof of the existence of the order: writers on the subject appear to assume the existence as self-evident or content themselves with proofs which seem unsatisfactory to the writer.

1. **The definition of the order.** We shall assume that $f(0) = a_0 = 0$, making this restriction with a view to introducing an operation H^α analogous to that by which Riemann defined integration and differentiation to incommensurable orders.§ For $\alpha > 0$ H^α has the simple expression

$$s(z) = H^\alpha f(z) = \int_0^1 \frac{v(t,\alpha)}{t} f(zt)\, dt, \qquad v(t,\alpha) = \frac{1}{\Gamma(\alpha)} \left(\log \frac{1}{t}\right)^{\alpha-1}.$$

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* Presented to the American Mathematical Society, March 1, 1924.

† Hadamard, Thèse: Essai sur l'étude des fonctions données par leur developpement de Taylor, 1892; a paper of the same title, Journ. de Math., sér. 4, t. 8 (1892), pp. 101–186. Part. III is concerned with the study of the order of a function at a singular point. Reference may also be made to Hadamard's treatise, La série de Taylor et son prolongement analytique, Scientia Series (1905), Chap. 5, §§ 5—7, pp. 44–49.

‡ P. Dienes, Thèse: Essai sur les singularités des fonctions analytiques, Gauthier-Villars 1909; a paper of the same title, Journ. de Math., sér. 6, t. 5 (1909), pp. 327–413. P. Dienes, Leçons sur les singularités des fonctions analytiques, 1913. Encykl. d. Math. Wiss., Bd. II$_3$, Hft. 4, Bieberbach, Neuere Untersuchungen über Funktionen von komplexen Variabeln, pp. 488–490. Fabry, L'ordre des points singuliers d'une série de Taylor, Acta Math., vol. 36 (1912), pp. 69–104.

§ Riemann, Versuch einer allgemeinen Auffassung der Integration und Differentiation, Werke, S. 331–344.

For negative values of α the operation may be defined by the equation

$$H^\alpha f(z) = \sum_1^\infty \frac{a_n}{n^\alpha} z^n, \qquad -\infty < \alpha < +\infty,$$

where the power series converges interior to the unit circle; or it may be noted that for positive integral p, $\left(z\frac{d}{dz}\right)^p H^p f(z) = f(z)$, whence we may define $H^\alpha f(z)$ for negative α as

$$s(z) = H^\alpha f(z) = \left(z\frac{d}{dz}\right)^p H^{p+\alpha} f(z), \qquad p+\alpha > 0.$$

It is shown by Hadamard in his Thèse that $s(z)$ is analytic everywhere except at the singular points of $f(z)$ where it also has singularities.

Before we define the order of $f(z)$ at a point of the unit circle, it is necessary to introduce a concept in the theory of functions of a real variable, due to Hadamard. A real or complex function of the real variable θ is said to be of *écart fini* on the interval (A, B) if the two integrals $n\int_a^b f(\theta)\sin n\theta\, d\theta$, $n\int_a^b f(\theta)\cos n\theta\, d\theta$ are bounded for all integral n and for all a, b on $A \leqq \theta \leqq B$. It is clear that in this definition the two integrals may be replaced by $n\int_a^b f(\theta)e^{ni\theta}\, d\theta$, $n\int_a^b f(\theta)e^{-ni\theta}\, d\theta$ without changing its content.

We are now ready to define the order of the function $f(z) = a_1 z + a_2 z^2 + \cdots$ on an arc $A \leqq \theta \leqq B$ of the unit circle: this order is the number ω such that for all $\alpha > \omega$, $H^\alpha f(z) = s(z)$ has the properties

(a) $s(z)$ is finite and continuous on the arc;

(b) $s(z)$ is of *écart fini* considered as a function of θ on the arc;

and such that for $\alpha < \omega$ one or the other of the two properties is no longer true of $s(z)$. In case the arc is the circumference of the circle, (a) is to be considered as implying continuity on the whole circumference.

The order of $f(z)$ at a point on the circle of convergence may be defined as the limit of the orders on any set of arcs including the point as an interior point and decreasing indefinitely in length.

Clearly the definition of the order for an arc requires justification. The existence of a single ω must be shown to make the definition effective. To do this it is both necessary and sufficient to show that if a function $s(z)$ satisfying (a) and (b) is operated on by H^ε where ε is any positive quantity, the new function obtained also satisfies (a) and (b). The necessity

of the condition is obvious; its sufficiency becomes apparent when it is noticed that as a consequence of it the real numbers α are divided into two classes A and A', each number of A being less than every number of A'. The numbers of A' are those for which $H^{\alpha}f(z)$ satisfies (a) and (b), the numbers of A those for which $H^{\alpha}f(z)$ fails to satisfy one requirement or the other. The number ω which we have called the order then exists as the cut between the classes A and A', unless A or A' is an empty class, cases in which the order may be taken as $-\infty$ or $+\infty$ respectively.

2. **A critique of previous treatments.** The exact meaning of the condition (a) requires interpretation. In the treatments which we have cited there is no statement concerning the sense in which the words "finite and continuous on the arc" are to be understood. Does the phrase mean that the function $s(z)$ is finite and continuous in the whole sector defined by the arc; or that for certain methods of approach of the point z to a point on the arc $s(z)$ has a definite limiting value at each point of the arc and that the aggregate of these values defines a continuous function of θ on the arc? It is evident from the mode of handling requirement (a) that it is assumed to imply the first of these interpretations as a logical consequence.* In § 3 we shall state a simple definition from which continuity in the sector may be deduced.

A more serious objection to the treatments of the definition of order is the almost universal omission of proof of its existence. So far as we can ascertain, P. Dienes† is the only author who has given any proof and, it seems to us, his method is open to criticism. Dienes first establishes the existence of the order for the whole circle, then relies on the theorem:

If $f(z)$ is continuous and of écart fini on the arc (A, B), $f(z)$ may be replaced by the sum of two functions $f_1(z)+f_2(z)$ where the first, f_1, is analytic interior to the circle of convergence and $H^{\epsilon}f_1(z)$, $\epsilon>0$, is continuous and of écart fini on the entire circumference; and the second, f_2, is analytic on the arc (A, B).

The argument is concluded by pointing out that, since f_2 is analytic on (A, B), so is $H^{\epsilon}f_2(z)$, $\epsilon > 0$; and $H^{\epsilon}f_2$ is thus continuous and of *écart fini* on the arc. Now $f_2(z)$ is defined by the integral $\frac{1}{2\pi i}\int_{C_2}\frac{f(x)\,dx}{x-z}$ taken over a curve C_2 lying, except for its endpoints A and B, entirely in the circle of convergence and, with the arc (A, B), surrounding the origin; while $f_1(z)$ is the integral $\frac{1}{2\pi i}\int_{C_1}\frac{f(x)\,dx}{x-z}$ taken over the arc in question. Thus Cauchy's integral formula shows us that $f_1(z)+f_2(z)=f(z)$ interior

* Hadamard, Thèse, p. 69; P. Dienes, Leçons sur les singularités, etc., p. 14.

† P. Dienes, Leçons sur les singularités, etc., p. 15.

to $C_1 + C_2$ whence we conclude that $f_2(z)$ may be developed about the origin in a Taylor's series with unit radius of convergence, this property holding for both $f(z)$ and $f_1(z)$. If A and B were both ordinary points for $f_2(z)$, the function would be analytic everywhere in the extended plane, and hence would reduce to a constant. We could then recast the theorem, taking $f_2(z) =$ constant; the new theorem thus obtained is false unless $f(z)$ is finite, continuous, and of *écart fini* on the whole circumference, as is shown by the example $f(z) = 1/(1-z)$, (A, B) any arc not including $z = 1$. Thus, in general, A or B or both must be singular points for $f_2(z)$. Under these conditions it is not possible to conclude that $H^\epsilon f_2(z)$ is continuous and of *écart fini* on the arc (A, B) for positive ϵ without a more detailed argument than that which is employed above.

In § 4 we hope to give a careful proof of the existence of the order, thus placing the foundations of an interesting chapter of the theory of functions on the basis of demonstration rather than of intuition.

3. **Continuity on the circle of convergence.** In this section we shall take up the meaning of condition (a). A little reflection will show that, even if $f(z)$ is finite and continuous on an arc of the circle of convergence for special methods of approach to points of the arc, $f(z)$ need not be continuous in the sector determined by the arc. For example, the function $e^{-z^{-4}}$ has the property that if the real part of z is held fast and the imaginary part allowed to approach zero a continuous function of the real variable x is obtained, namely $e^{-x^{-4}}$. If the z-plane is transformed by a linear fractional transformation taking the real axis into the unit circle, the transform of $e^{-z^{-4}}$ is a function finite and continuous on the whole unit circle for special methods of approach to the circumference, but not in the whole circle, because it has an essential singularity on the circumference of such a character as to make continuity impossible.

From the point of view of the study of a function by means of its Taylor's series it is desirable to make the criteria for continuity in a sector as simple as possible. In particular, it is useful to make the discussion depend on the behavior of the power series for simple methods of approach of the variable z to values on the arc of the sector. It is with this idea in mind that we set up the test for continuity developed in our first theorem.

We note first of all three lemmas, for which proofs are to be found in the literature.

LEMMA I. *Any sector, inclusive of its boundary points, can be mapped in a one-to-one manner and continuously on a circle, the transformation being conformal except at the three vertices of the sector. The Green's function for the sector and its conjugate are harmonic except at the three vertices of*

*the sector and at one interior point, and are continuous in the whole sector except at this interior point.**

LEMMA II. *If $u(x, y)$, bounded and harmonic interior to a sector, approaches the value u_0 when the point $P:(x, y)$ approaches a point P_0 of the boundary, other than one of the three vertices, along a path meeting the boundary orthogonally with finite curvature in P_0, then $u(x, y)$ takes on the value u_0 when P approaches P_0 along any other path of the same character.*†

LEMMA III. *If $u(x, y)$ is bounded and harmonic interior to a circle and takes on continuous boundary values for radial approach to the boundary, there exists no other function bounded and harmonic interior to the circle which, for radial approach to the boundary, takes on the same boundary values except possibly at a set of points of measure zero.*‡

We may now state and prove the

THEOREM I. *If at every point of the region T' obtained by deleting from a sector T its bounding circular arc $f(z)$ is defined and satisfies the requirements that*

(a) *$f(z)$ is analytic in T';*

(b) *$f(z)$ is bounded in the neighborhood of each point of the arc;*

(c) *$f(z)$ approaches a limit as z approaches the bounding arc along a radius and the totality of these limits defines a continuous function along the arc;*

then $f(z)$ is continuous in T if it is defined along the arc by its limiting values.

PROOF. We are enabled by (b) to show that $f(z)$ is bounded in T'. About each point of the arc we can construct a circle such that for all points of T' in this circle $f(z)$ is bounded; but by the Heine-Borel theorem a finite chain of these circles can be found completely covering the arc. In particular, the circles may be taken as overlapping. It then follows without difficulty that $f(z)$ is bounded in T'.

We denote by $u(x, y)$ the real part of $f(z)$. Concerning this function we know that it is bounded and harmonic in T' and that, when the point $P:(x, y)$ approaches any point of the bounding arc of T along a radius, $u(x, y)$ has a limit u. By Lemma II we can let P approach a point on the arc along any curve meeting the arc orthogonally with finite curvature, the end-points of the arc being excluded. The curves $H(x, y) =$ constant, where $H(x, y)$ is the negative of the conjugate of the Green's function

* Osgood, Lehrbuch der Funktionentheorie, 2. Aufl. (1912), Kap. 14, §§ 1–4.

† Kellogg, Harmonic functions and Green's integral, Trans. Amer. Math. Soc., vol. 13 (1912), pp. 109–132, Theorem X.

‡ Fatou, Séries trigonométriques et séries de Taylor, Acta Math. vol. 30 (1905–1906), pp. 335–400; see p. 349.

for T, may be used. To assure ourselves that the courves $H =$ constant satisfy the conditions of Lemma II, we observe that in accordance with Lemma I both the Green's function and its conjugate are harmonic in T except at its vertices and one interior point. Since the boundary of T is obtained by setting the Green's function equal to zero, any curve $H =$ constant meeting it in a point not a vertex meets it orthogomally; further at such a point $H(x, y)$ is harmonic with continuous derivatives of all orders, so that the curve $H =$ constant meets the boundary with finite curvature. Finally the boundary values of $u(x, y)$ along the arc and its given values along the bounding radii of T define a continuous function on the boundary.

We may now use Lemma I to map T on the unit circle. Corresponding to $u(x, y)$ we obtain a function bounded and harmonic interior to the circle and taking on continuous boundary values for radial approach to the circumference, except along the radii determined by the transforms of the vertices of T; for the curves $H =$ constant are taken into the radii of the circle if the transformation is properly determined. Lemma III now shows that the function thus obtained can be none other than the function defined by Poisson's integral for the continuous boundary values, and it is therefore continuous in the circle. Returning now to the sector T, we see that $u(x, y)$ is continuous in T.

The same reasoning applies to the function $v(x, y)$ where $iv(x, y)$ is the pure imaginary part of $f(z)$.

It follows that $f(z)$ is continuous in T if properly defined on the arc. In case the sector T is the whole circle we need only Lemma III to obtain this result.

We shall henceforth interpret the statement that $f(z)$ is finite and continuous on the arc (A, B) of the unit circle to mean that $f(z)$, given as analytic interior to the circle, satisfies conditions (b) and (c) of Theorem I in the sector defined by the arc. We shall then be able to use the fact that $f(z)$ is continuous in the whole sector.

4. **Existence of the order.** Let us suppose that we have operated on the function

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

with the operation H^α, obtaining the new function

$$s(z) = a_1 z + 2^{-\alpha} a_2 z^2 + 3^{-\alpha} a_3 z^3 + \cdots = z S(z).$$

The function $s(z)$ is assumed to be finite, continuous, and of *écart fini* on an arc of the unit circle, interior to which it is known to be analytic.

We wish to show that the function $H^\varepsilon s(z)$ is also finite, continuous, and of *écart fini* on the arc for all positive ε. From § 1 we have

$$s_1(z) = H^\varepsilon s(z) = \int_0^1 \frac{v(t,\varepsilon)}{t} s(zt)\,dt = z\int_0^1 v(t,\varepsilon)\,S(zt)\,dt = zS_1(z)$$

It is now an easy matter to prove

THEOREM II. *If $s(z)$ is finite and continuous on an arc of the unit circle, then so is $H^\varepsilon s(z)$ for all positive ε.*

PROOF. From § 3 it turns out that $s(z)$ and also $S(z) = s(z)/z$ are continuous in the sector T defined by the arc. Thus

$$|S_1(z'') - S_1(z')| \leqq \int_0^1 v(t,\varepsilon)\,|S(z''t) - S(z't)|\,dt < \eta\int_0^1 v(t,\varepsilon)\,dt = \eta$$

if $|z'' - z'| < \delta$ for z'', z' in T, since by the continuity of $S(z)$

$$|S(z''t) - S(z't)| < \eta \qquad |tz'' - tz'| \leqq |z'' - z'| < \delta.$$

Hence $S_1(z)$ and $s_1(z) = zS_1(z)$ are both continuous in T, as we were to prove.

In order to show that $s_1(z)$ is also of *écart fini* on the arc, we must use less obvious means. Our task is somewhat simplified if we obtain the preparatory

LEMMA IV. *If $f(z)$ is defined and of écart fini on an arc of the unit circle, then $zf(z)$ and $z^{-1}f(z)$ are also of écart fini on the arc.*

PROOF. By definition $n\int_a^b f(e^{i\theta})\,e^{\pm ni\theta}\,d\theta$ is bounded for all integral n and for all a and b on the arc $A \leqq \theta \leqq B$. To show that the definition of a function of *écart fini* applies to $zf(z)$ and $z^{-1}f(z)$ we consider four equations of which one will serve as a type

$$n\int_a^b e^{i\theta} f(e^{i\theta})\,e^{ni\theta}\,d\theta = (n+1)\int_a^b f(e^{i\theta})\,e^{(n+1)i\theta}\,d\theta - \int_a^b f(e^{i\theta})\,e^{(n+1)i\theta}\,d\theta.$$

The desired result is an obvious consequence of this equation and its analogues.

We are now ready to proceed to

THEOREM III. *If $s(z)$ is finite, continuous, and of écart fini on an arc of the unit circle, then so is $H^\varepsilon s(z)$ for all positive ε.*

PROOF. By theorem II, $s_1(z) = H^\epsilon s(z)$ and $s(z)$ are continuous in the whole sector T with arc $A \leqq \theta \leqq B$. If $s(z)$ is of *écart fini* on the arc, then Lemma IV shows that $S(z) = z^{-1} s(z)$ is also of *écart fini* on the arc; to show that $s_1(z) = z S_1(z)$ is of the same character, it will be sufficient, by Lemma IV, to show that the integrals $J_1 = n \int_a^b S_1(e^{i\theta})\, e^{ni\theta}\, d\theta$ and $J_2 = n \int_a^b S_1(e^{i\theta})\, e^{-ni\theta}\, d\theta$ are bounded for all positive integral n and for all a and b on $A \leqq \theta \leqq B$.

J_1 is easily discussed by integrating the function $\frac{n}{i} z^{n-1} S_1(z)$ around the boundary of the sector of the unit circle included by the radii $\theta = a$ and $\theta = b$. By Cauchy's integral theorem the integral about the contour vanishes; this integral may be written, however, as the sum of three integrals I_1, I_2, I_3, along the arc (a, b) and the two radii respectively. Thus $I_1 + I_2 + I_3 = 0$ whence $|I_1| \leqq |I_2| + |I_3|$. Now I_1, except possibly for sign, is precisely the integral J_1; and by the fact that $|e^{ni\theta} S_1(r e^{i\theta})| < m$ throughout T it is readily shown that $|I_2|$ and $|I_3|$ are each less than $m \int_0^1 n r^{n-1}\, dr = m$. Hence $|J_1| = |I_1| < 2m$ for all positive integral n and for all a and b on $A \leqq \theta \leqq B$.

The second integral J_2 is considered through the equation

$$S_1(z) = \int_0^1 v(t, \epsilon)\, S(zt)\, dt.$$

Since $S(z)$ is continuous in T, the integral defining $S_1(z)$ is absolutely and uniformly convergent for all z in the sector; it is therefore permissible to integrate under the integral sign. In this way we find

$$J_2 = \int_0^1 v(t, \epsilon) \left[n \int_a^b S(te^{i\theta}) e^{-ni\theta}\, d\theta \right] dt = \int_0^1 v(t, \epsilon)\, J_2'(t)\, dt.$$

By integrating the function $n S(z)/i z^{n+1}$ around a suitable contour we find an upper bound for $J_2'(t)$. This contour is to be the boundary of the region included by the two circles $r = 1$ and $r = t$, $0 < t \leqq 1$, and the two radii $\theta = a$ and $\theta = b$. Applying Cauchy's integral theorem we see that the sum of the integrals I_1, I_2, I_3, I_4, over the four parts of the boundary respectively, vanishes. The integral I_1 over the arc of the circle $r = t$ between $\theta = a$ and $\theta = b$ is, except possibly for sign, precisely the integral $t^{-n} J_2'(t)$. A similar relation holds between I_3 taken over the arc (a, b) of the unit circle and the integral $n \int_a^b S(e^{i\theta}) e^{-ni\theta}\, d\theta$; but *by hypothesis* the

last integral is bounded for all positive integral n and for all a and b on $A \leqq \theta \leqq B$. The remaining integrals I_2 and I_4 are each less in absolute value than $m' \int_t^1 \frac{n\,dr}{r^{n+1}} = m'\left(\frac{1-t^n}{t^n}\right)$ since in the entire sector T $|e^{-ni\theta} S(re^{i\theta})| < m'$. We see then that

$$|J_2'(t)| \leqq t^n|I_2| + t^n|I_3| + t^n|I_4| < 2m'(1-t^n) + m''t^n < 2m' + m'', \quad 0 < t \leqq 1,$$
$$|J_2'(0)| = |S(0)(e^{-nia} - e^{-nib})|,$$

or $|J_2'(t)| < M$ for all positive integral n, for all a and b on $A \leqq \theta \leqq B$, and for all t, $0 \leqq t \leqq 1$. Finally

$$|J_2| \leqq \int_0^1 v(t, \epsilon)|J_2'(t)|\,dt < M \int_0^1 v(t, \epsilon)\,dt = M.$$

With this inequality the proof that $S_1(z)$ and $s_1(z)$ are of *écart fini* on the arc (A, B) is brought to a close.

As we pointed out at the end of § 1, the existence of the order of the function $f(z)$ on the arc is an immediate consequence of Theorem III. It is also apparent that the order on an arc cannot be less than the order on any component arc. From this fact follows the existence of the order of $f(z)$ at a point P of the circle of convergence, as defined in § 1.

If there is given an infinite set of arcs $A_1, A_2, A_3, \cdots$ on the unit circle where each arc of the set includes P as an interior point and the measures or lengths of the arcs form a sequence with limit zero, $f(z)$ has order α_k on A_k; and we wish to show that $\lim_{k=\infty} \alpha_k$ exists and is equal to the corresponding limit for any other infinite set of arcs $[B_k]$ enjoying the same properties as the set $[A_k]$. We include $+\infty$ and $-\infty$ in the set of real numbers for this discussion. We consider the two sequences of point sets defined by the equations*

$$C_k' = A_1 B_1 A_2 B_2 \cdots A_k B_k \qquad (k = 1, 2, 3, \cdots),$$
$$C_k'' = A_k + B_k + A_{k+1} + B_{k+1} + \cdots \qquad (k = 1, 2, 3, \cdots).$$

The set of points C_k' is clearly an arc and cannot reduce to a point since the point P is an interior point of $A_1, A_2, \cdots, A_k$; $B_1, B_2, \cdots, B_k$. The set of the points C_k'' is likewise an arc. For any k we can determine an n_k such that, for all $n > n_k$, $A_n < A_k$ and $B_n < B_k$; thus

$$C_k'' = A_k + B_k + A_{k+1} + B_{k+1} + \cdots + A_{n_k} + B_{n_k}$$

* By the sum or product of two sets of points we mean the logical sum or product; cf. de la Vallée Poussin, Intégrales de Lebesgue, Paris 1916, p. 6.

so that C_k'' is the sum of a finite set of arcs with a common interior point and is therefore an arc. By the manner of definition of the sets of arcs $[C_k']$ and $[C_k'']$ we can write down the following facts at once:

$$C_k' \leqq A_k \leqq C_k'', \qquad \gamma_k' \leqq \alpha_k \leqq \gamma_k'', \qquad C_{k+1}' \leqq C_k', \qquad \gamma_{k+1}' \leqq \gamma_k',$$
$$C_k' \leqq B_k \leqq C_k'', \qquad \gamma_k' \leqq \beta_k \leqq \gamma_k'', \qquad C_{k+1}'' \leqq C_k'', \qquad \gamma_{k+1}'' \leqq \gamma_k'',$$
$$m(C_k'') < 2\varepsilon,\ k > K \text{ if } m(A_k) < \varepsilon,\ k > K; \text{ and } m(B_k) < \varepsilon,\ k > K,$$

where we denote by the Greek letters β, γ the orders of $f(z)$ on the arcs denoted by the corresponding Roman letters B, C respectively. Because the sequences $[\gamma_k']$ and $[\gamma_k'']$ are never-increasing, they have unique limits γ' and γ'' respectively; and $\gamma' \leqq \gamma''$ by the inequalities above. Now for any k we can find an n_k such that $C_{n_k}'' \leqq C_k'$ and $\gamma_{n_k}'' \leqq \gamma_k'$; in the limit we obtain $\gamma'' \leqq \gamma'$. Thus $\gamma' = \gamma''$ and in consequence the two sequences $[\alpha_k]$ and $[\beta_k]$ have the limit $\gamma' = \gamma''$. This limit is the order of $f(z)$ at the point P.

The foundations of the theory of the order of an analytic function at points on the circle of convergence of its Taylor's series have been examined in the preceding paragraphs. For information concerning the further developments and applications of the concept of order the reader is referred to the literature which we have cited above.

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REPRESENTATIONS OF INTEGERS IN CERTAIN BINARY, TERNARY, QUATERNARY AND QUINARY QUADRATIC FORMS AND ALLIED CLASS NUMBER RELATIONS.

By E. T. Bell.

1. **Introduction.** When there is but one class in the principal genus of binary quadratic forms of a given negative determinant, the number of representations of an integer in the principal form can be found at once in terms of the divisors of the integer. For certain ternary forms there are equally simple theorems, the divisor functions being replaced in this case by class numbers. The analogous situation for quaternary forms again involves functions of divisors. For quinary forms the like exists in numerous instances if the integer represented is the square of an odd number times a power of 2. These theorems yield class number relations. Those deduced in this way from quinary forms are of a wholly new type which is of particular interest because its connection, if any, with elliptic functions is remote. The types of relation change with the number of indeterminates in the forms from which they are derived. Conversely, any such class number relation can be interpreted as a theorem on numbers of representations in each of several quadratic forms. We shall give only a few of the relations. These are sufficient to show clearly the method of finding all. Extensive lists of new class number relations can easily be written out by applying the principles developed here to the results given in papers cited later.

2. **Notation.** $N(k = a'x^2 + b'y^2 + \cdots + e'u^2)$ is used to denote the number of representations of the integer $k \geqq 0$ in the form $a'x^2 + b'y^2 + \cdots + e'y^2$, where $a', b', \cdots e'$ are constant integers. Every integer k is of the form $p^\alpha n (\alpha \geqq 0)$ where p is prime and n is prime to p. Let $\sigma, \alpha, \beta, \gamma, \cdots, \varepsilon$ denote integers $\geqq 0$, and $a, b, c, \cdots, e$ integers $\not\equiv 0$ prime to the prime p having no common divisor > 1. Since $p^\alpha n, \cdots, ep^\varepsilon$ are arbitrary integers, there is no loss of generality in restricting the discussion to $N(p^\sigma n = ap^\alpha x^2 + bp^\beta y^2 + cp^\gamma z^2 + \cdots + ep^\varepsilon u^2)$, or, as it may conveniently be written,

$$(\sigma;\ \alpha,\ \beta,\ \gamma,\ \cdots,\ \varepsilon). \tag{1}$$

Throughout the paper m is an odd integer, k an arbitrary integer.

3. **Canonical form.** The case of (1) in which $\alpha > 0$ is immediately reduced to that in which $\alpha = 0$. By the definition of $\beta, \gamma, \cdots$, an inequality such as $\gamma \geqq \beta$ can be eliminated by replacing γ by $\gamma + \beta$. Evidently there is no loss of generality in assuming that each of $\beta, \gamma, \cdots, \varepsilon$ in (1)

is not less than its immediate predecessor. Hence finally it is sufficient to discuss the canonical form

$$(2) \qquad (\sigma;\ 0,\ \beta,\ \beta+\gamma,\ \beta+\gamma+\delta,\ \cdots,\ \beta+\gamma+\delta+\cdots+\varepsilon).$$

This symbol is subject to a species of reduction, which has been fully developed in a forthcoming paper.*

By means of this, (2) can be replaced by

$$(2.1) \qquad (\sigma';\ \alpha',\ \beta',\ \beta'+\gamma',\ \cdots,\ \beta'+\gamma'+\cdots+\varepsilon')$$

in which $\alpha' = 0$ or 1, and for some choice of the signs $=, <$, $\sigma' \leqq \sigma$, $\beta' \leqq \beta, \cdots, \beta'+\gamma'+\cdots+\varepsilon' \leqq \beta+\gamma+\cdots+\varepsilon$, and no further reduction of this sort is possible. We shall write down only the final formulas from the paper cited as required.

4. **Reduction of** $(\sigma;\ 0,\ \beta)$. The complete reduction by means of § 3 is

$$\begin{aligned}
&(3.1) \quad \alpha \leqq \beta, \quad (2\alpha;\ 0,\ 2\beta) && = (0;\ 0,\ 2\beta-2\alpha);\\
&(3.2) \quad \alpha < \beta, \quad (2\alpha+1;\ 0,\ 2\beta) && = 0;\\
&(3.3) \quad\ \ \ \ \ \ (2\beta+\alpha+1;\ 0,\ 2\beta) && = (\alpha+1;\ 0,\ 0);\\
&(4.1) \quad \alpha \leqq \beta, \quad (2\alpha;\ 0,\ 2\beta+1) && = (0;\ 0,\ 2\beta+1-2\alpha);\\
&(4.2) \quad \alpha < \beta, \quad (2\alpha+1;\ 0,\ 2\beta+1) && = 0;\\
&(4.3) \quad\ \ \ \ \ \ (2\beta+\alpha+1;\ 0,\ 2\beta+1) && = (0;\ 0,\ 1).
\end{aligned}$$

Thus, for example, from (3.3) when either the number of representations of $p^{2\beta+\alpha+1}n$ (n prime to the prime p, $\alpha, \beta \geqq 0$, by § 2) in the form $ax^2+bp^{2\beta}y^2$ (a, b, co-prime and prime to p), or the number of representations of $p^{\alpha+1}n$ in the form ax^2+by^2 is known, so also is the other.

5. **Representations in** x^2+Dy^2, $D>0$. For very few values of D have complete evaluations of $N(k = x^2+Dy^2)$ been found. All those in the literature are immediate consequences of a theorem due to Dirichlet:†

"If n is prime to 2Δ, and if $\tau = 1, 2$, the total number of representations of τn *by the ensemble of a complete system of representative primitive forms of determinant* Δ, properly or improperly primitive according as $\tau = 1$ or $\tau = 2$, is $\varrho\sum(\Delta|\delta)$, the $\sum$ extending to all divisors δ of n, where $(\Delta|\delta)$ is the Legendre-Jacobi symbol, $\varrho = 1$ if $\Delta > 0$, $= 4$ if $\Delta = -1$, $= 6$ if $\Delta = -3$, and otherwise $\tau = \varrho = 2$."

* Reductions of enumerations in homogeneous forms, Bull. Amer. Math. Soc., vol. 30 (1924), pp. 345–351.

† Zahlentheorie, 4th ed. § 91; cf. also Dickson, History of the Theory of Numbers, vol. 3, p. 19.

The root of the difficulty is in the italcized clause. Obviously if there is but one genus for the determinant $-D$, or if the principal genus contains but one class, or if in each order there is but one class, Dirichlet's theorem, combined with § 4 and other equally elementary considerations when D is composite, yields an evaluation of $N(k = x^2 + Dy^2)$. But unless one of these conditions is satisfied, it does not seem easy to obtain a complete solution by the Gauss theory of binary quadratic forms alone.

As an illustration, consider $N(k = x^2 + 7y^2)$. An application of § 4 disposes of the case in which k is divisible by 7, reducing it to that in which k is prime to 7. For the determinant -7 there is but one properly primitive class and but one improperly primitive, represented respectively by $x + 7y^2$, $2x^2 + 2xy + 4y^2$. By Dirichlet's theorem and § 4 we find that the number of representations of an odd number m in the form $x^2 + 7y^2$ is twice the excess of the number of divisors of m of the forms $14k + 1, 9, 11$ over the number of $14k + 3, 5, 13$ divisors. The complete evaluation of $N(2^\alpha 7^\beta = x^2 + 2y^2)$ can be obtained by combining this result with those given by the modular equation for the transformation of order 7 in elliptic functions.

The negative determinants $\geqq -100$ for which simple results can be found as indicated are the negatives of

2,	3,	4,	5,	6,	7,	8,	9,	10,	12,
13,	15,	16,	18,	21,	22,	24,	25,	28,	30,
33,	37,	40,	42,	45,	48,	57,	58,	60,	70,
72,	78,	85,	88,	93,					

as may be seen by inspection of Cayley's tables of generic characters.*

It is unnecessary here to consider any determinants other than -2^α, as the corresponding information which we shall require from ternary forms is lacking for all others.

6. **Determinants -2^α.** Let $\xi(k)$ denote the excess of the number of $4t + 1$ divisors of k over the number of $4t + 3$ divisors, and $\omega(k)$ the excess of the number of $8t + 1, 3$ divisors of k over the number of $8k + 5, 7$ divisors. Then, as is well known, if $k = 2^\alpha m$, $\alpha \geqq 0$, (m odd by § 2), we have

$$N(2^\alpha m = x^2 + y^2) = 4\xi(2^\alpha m) = 4\xi(m); \tag{5}$$

$$N(2^\alpha m = x^2 + 2y^2) = 2\omega(2^\alpha m) = 2\omega(m). \tag{6}$$

* Collected Papers, vol. 5, pp. 144-147. The following errata in these tables are significant in the present connection: for $\Delta = -17$ delete the long bar; for $\Delta = -40$ insert a short bar under each of (0,40), (2,11), (0,8), (3,7), supply $+ - + -$ in the δ column and $+ - - +$ in the ε column, delete the signs in the $\delta\varepsilon$ column; for $\Delta = -88$ insert a short bar under each of (0,88), (2,23), (0,11), (4,13), supply $+ - - +$ in the δ column.

There is but one class for the determinant -4. Hence Dirichlet's theorem combined with the properties of $(\Delta \mid \delta)$ and § 4 gives*

$$N(m = x^2+4y^2) = 2\xi(m); \tag{7.1}$$

$$N(2m = x^2+4y^2) = 0; \tag{7.2}$$

$$N(2^{\alpha+2}m = x^2+4y^2) = 4\xi(m). \tag{7.3}$$

All odd numbers represented by x^2+8y^2 are of the form $8t+1$. As before we get

$$m = 8t+1, \qquad N(m = x^2+8y^2) = \omega(m); \tag{8.1}$$

$$N(2m = x^2+8y^2) = 0; \tag{8.2}$$

$$N(2^{\alpha+2}m = x^2+8y^2) = 2\omega(m). \tag{8.3}$$

As always, if no condition is explicitly indicated, m is an arbitrary odd integer >0.

Similarly, for the determinant -16,

$$m = 8t+1, \qquad N(m = x^2+16y^2) = 2\xi(m); \tag{9.1}$$

$$N(2^\gamma m = x^2+16y^2) = 0, \qquad \gamma = 1, 3; \tag{9.2}$$

$$N(4m = x^2+16y^2) = 2\xi(m); \tag{9.3}$$

$$N(2^{\alpha+4}m = x^2+16y^2) = 4\xi(m). \tag{9.4}$$

Each of the enumerations for the determinants -1, -2, -4, -8, -16 is complete. Continuing in the same way we find the following *incomplete* enumerations.

$$N(2^\gamma m = x^2+32y^2) = 0, \qquad \gamma = 1, 3; \tag{10.1}$$

$$m = 8t+1, \qquad N(4m = x^2+32y^2) = 2\omega(m); \tag{10.2}$$

$$N(2^{\alpha+4}m = x^2+32y^2) = 2\omega(m); \tag{10.3}$$

and for the determinant -64,

$$N(2^\gamma m = x^2+64y^2) = 0, \qquad \gamma = 1, 3, 5; \tag{11.1}$$

$$N(16m = x^2+64y^2) = 2\xi(m); \tag{11.2}$$

$$N(2^{\alpha+6}m = x^2+64y^2) = 4\xi(m); \tag{11.3}$$

* The number of representations of $2^{\alpha+2}m$ in this form is erroneously stated by Pepin, Journ. des Math., sér. 4, vol. 6 (1890) p. 11, as half that given here. I have not examined whether this oversight affects any subsequent theorems of Pepin's memoir.

while for -128 the function ω reappears,

$$(12.1) \qquad N(2^\gamma m = x^2+128y^2) = 0, \qquad \gamma = 1, 3, 5;$$

$$(12.2) \qquad m = 8t+1, \qquad N(16m = x^2+128y^2) = 2\omega(m);$$

$$(13.3) \qquad N(2^{\alpha+6}m = x^2+128y^2) = 2\omega(m),$$

and so on indefinitely, the functions ξ, ω alternating.

7. Connection with binary quadratic class numbers. The connection is made thus. For certain ternary forms $ax^2+by^2+cz^2$, discussed in § 8, the total number of representations can be expressed in terms of $F(k)$ or $E(k) \equiv F(k) - F_1(k)$, where $F(k)$, $F_1(k)$ are respectively the numbers of odd, even classes of binary quadratic forms, with the usual conventions, of negative determinant $-k$. It is important to notice that zero has precisely one representation in any homogeneous form with coefficients all >0. This remark must be borne in mind in assigning conventional values for zero arguments to functions $\xi, \omega, F, E, \cdots$ giving numbers of representations in specific forms. Write

$$N(k = x^2+By^2+Cz^2) = N(k), \qquad N(k = x^2+By^2) = N_1(k),$$
$$N(k = x^2+Cy^2) = N_2(k).$$

Then, the summation extending to all integers $s \gtreqless 0$ which render the arguments of the summands $\geqq 0$, we have

$$(14) \qquad N(k) = \sum N_1(k - Cs^2) = \sum N_2(k - Bs^2).$$

8. Representations in $x^2+By^2+Cz^2$. Complete enumerations for given values of B, C are rare. Disregarding an unproved statement of Liouville for the case $(B, C) = (2, 3)$ we have *complete* enumerations in the following ten cases only:

$$(B, C) = (1, 1), \quad (1, 2), \quad (1, 4), \quad (1, 8), \quad (2, 2),$$
$$(2, 4), \quad (2, 8), \quad (4, 4), \quad *(4, 8), \quad (8, 8)$$

of which (1, 1) is the classic theorem of Gauss,

$$(15) \qquad N(k = x^2+y^2+z^2) = 12E(k),$$

and the rest are obtained in another paper.*

Even partial enumerations are lacking for other values of (B, C).† It will be sufficient here to quote two results from the forthcoming paper.

* To be published in the American Mathematical Monthly.

† Cf. Dickson's History, vol. 2, chap. 7; vol. 3, chap. 9.

For $(B, C) = (1, 2)$ we found

$$N(2k+1 = x^2 + y^2 + 2z^2) = 4F(4k+2); \tag{16.1}$$

$$N(2k = x^2 + y^2 + 2z^2) = 12E(k), \tag{16.2}$$

and for $(B, C) = (2, 2)$,

$$N(4k+1 = x^2 + 2y^2 + 2z^2) = 4F(4k+1); \tag{17.1}$$

$$N(4k+2 = x^2 + 2y^2 + 2z^2) = 4F(4k+2); \tag{17.2}$$

$$N(8k+3 = x^2 + 2y^2 + 2z^2) = 8F(8k+3); \tag{17.3}$$

$$N(8k+7 = x^2 + 2y^2 + 2z^2) = 0; \tag{17.4}$$

$$N(2^{\alpha+2}m = x^2 + 2y^2 + 2z^2) = 12E(2^\alpha m). \tag{17.5}$$

9. **Class number formulas from ternary forms.** Applying (14) to (16.1), (16.2) we get by means of (5), (6):

$$F(4k+2) = \sum \xi(2k+1-2s^2); \tag{18.1}$$

$$2F(4k+2) = \sum \omega(2k+1-s^2); \tag{18.2}$$

$$3E(k) = \sum \xi(k-s^2); \tag{18.3}$$

$$6E(k) = \sum \omega(2k-s^2), \tag{18.4}$$

in which, also in the following, *by convention*

$$\xi(0) = \tfrac{1}{4}, \qquad \omega(0) = \tfrac{1}{2}, \tag{19}$$

and the $\sum$ is as in § 7.

Similarly from (17.1)–(17.5) we find

$$2F(4k+1) = \sum \omega(4k+1-2s^2); \tag{20.1}$$

$$4F(8k+3) = \sum \omega(8k+3-2s^2). \tag{20.2}$$

Without taking space to give the equally simple calculations, which the reader may easily supply on referring to the papers cited and proceeding as above, we may state that the complete set of relations obtainable from the ten forms in § 8 is as follows:

$$(21)\quad 4F(4k+1) = 2\sum\xi(4k+1-4s^2) = 4\sum\xi(4k+1-t^2),$$
$$= 2\sum\omega(4k+1-2s^2) = \sum\omega(8k+2-t^2);$$

$$(22)\quad 2F(4k+2) = 2\sum\xi(2k+1-2s^2) = \sum\xi(4k+2-t^2),$$
$$= 2\sum\omega(2k+1-4s^2) = \sum\omega(2k+1-s^2) = 2\sum\omega(2k+1-t^2);$$

$$(23)\quad 4F(8k+3) = \sum\omega(8k+3-2s^2) = 2\sum\omega(8k+3-2t^2);$$

$$(24)\quad 6E(k) = 2\sum\xi(k-s^2) = \sum\omega(2k-s^2);$$

$$(25)\quad 2F(8k+1) = \sum\omega(8k+1-8s^2),$$

in which the summations refer to all integers $s \gtrless 0$ or to all odd integers $t \gtrless 0$ rendering the arguments of the summands $\geqq 0$. The conventions (19) hold in the above, and in the following recurrences for ξ, ω obtained from (21)–(25):

$$(26)\quad \sum[3(-1)^s-1]\,\xi(4k+1-s^2) = 0;$$

$$(27)\quad \sum\xi(8k+2-t^2)-2\sum\xi(4k+1-8s^2) = 0;$$

$$(28)\quad \sum\xi(8k+6-t^2)-2\sum\xi(4k+3-2t^2) = 0;$$

$$(29)\quad \sum(-1)^s\omega(8k+3-2s^2) = 0;$$

$$(30)\quad \sum(-1)^s\omega(2k+1-s^2) = 0;$$

$$(31)\quad \sum\omega(8k+2-t^2)-2\sum\omega(4k+1-2s^2) = 0;$$

$$(32)\quad 2\sum\xi(k-s^2) = \sum\omega(2k-s^2).$$

These have been checked numerically. From the way in which (26)–(32 were derived this is a sufficient numerical verification of (21)–(25), which, however, were checked independently.

10. **Class number relations from quaternary forms.** We shall pass over these with a bare mention, as all of the necessary information for writing them out, if desired, is readily available in the papers of Liouville and Pepin* in conjunction with the complete set of results cited in § 8. The formula corresponding to (14) is in this case

$$(33)\quad N(k = x^2+By^2+Cz^2+Du^2) = \sum N_1(k-Ds^2),$$

where $N_1(k) \equiv N(k = x^2+By^2+Cz^2)$.

* Journ. des Math., (2), 7 (1862); (2), 8 (1863); (2), 10 (1865) (several papers); ibid., (4), 6 (1890), pp. 1-67.

11. Quinary reduction formulas. As indicated in § 3 we find at once the complete set of reduction formulas for

$$(34) \qquad (\sigma;\ 0,\ \beta,\ \beta+\gamma,\ \beta+\gamma+\delta,\ \beta+\gamma+\delta+\varepsilon).$$

It is convenient to separate cases, first according to even or odd β, and then each of these according to even or odd γ. In all it is necessary further to distinguish odd and even σ. All cases are included in (35), (36), (37).

$$(35) \qquad N_1(p^\sigma n) \equiv (\sigma;\ 0,\ 2\beta,\ 2\beta+\gamma,\ 2\beta+\gamma+\delta,\ 2\beta+\gamma+\delta+\varepsilon):$$

$$\alpha \leqq \beta, \quad N_1(p^{2\alpha} n) = (0;\ 0,\ 2\beta-2\alpha,\ 2\beta+\gamma-2\alpha,\ 2\beta+\gamma+\delta-2\alpha,\ 2\beta+\gamma+\delta+\varepsilon-2\alpha);$$

$$\alpha < \beta, \quad N_1(p^{2\alpha+1} n) = 0;$$

$$N_1(p^{2\beta+\alpha} n) = (\alpha;\ 0,\ 0,\ \gamma,\ \gamma+\delta,\ \gamma+\delta+\varepsilon).$$

These dispose of (34) when in (34) β is even. The first case of (34) with β odd is that in which γ is even. For this we have

$$(36) \qquad N_2(p^\sigma n) \equiv (\sigma;\ 0,\ 2\beta+1,\ 2\beta+2\gamma+1,\ 2\beta+2\gamma+\delta+1,\ 2\beta+2\gamma+\delta+\varepsilon+1):$$

$$\alpha \leqq \beta, \quad N_2(p^{2\alpha} n) = (0;\ 0,\ 2\beta+1-2\alpha,\ 2\beta+2\gamma+1-2\alpha,\ 2\beta+2\gamma+\delta+1-2\alpha,\ 2\beta+2\gamma+\delta+\varepsilon+1-2\alpha);$$

$$\alpha < \beta, \quad N_2(p^{2\alpha+1} n) = 0;$$

$$\alpha \leqq \gamma, \quad N_2(p^{2\beta+2\alpha} n) = (0;\ 0,\ 1,\ 2\gamma+1-2\alpha,\ 2\gamma+\delta+1-2\alpha,\ 2\gamma+\delta+\varepsilon+1-2\alpha);$$

$$N_2(p^{2\beta+2\gamma+2\alpha+2} n) = (2\alpha+1;\ 1,\ 0,\ 0,\ \delta,\ \delta+\varepsilon);$$

$$\alpha \leqq \gamma, \quad N_2(p^{2\beta+2\alpha+1} n) = (0;\ 1,\ 0,\ 2\gamma-2\alpha,\ 2\gamma+\delta-2\alpha,\ 2\gamma+\delta+\varepsilon-2\alpha);$$

$$N_2(p^{2\beta+2\gamma+2\alpha+1} n) = (2\alpha;\ 1,\ 0,\ 0,\ \delta,\ \delta+\varepsilon).$$

Finally when in (34) both β, γ are odd we get

$$(37) \qquad N_3(p^\sigma n) \equiv (\sigma;\ 0,\ 2\beta+1,\ 2\beta+2\gamma+2,\ 2\beta+2\gamma+\delta+2,\ 2\beta+2\gamma+\delta+\varepsilon+2):$$

$$\alpha \leqq \beta, \quad N_3(p^{2\alpha} n) = (0;\ 0,\ 2\beta+1-2\alpha,\ 2\beta+2\gamma+2-2\alpha,\ 2\beta+2\gamma+\delta+2-2\alpha,\ 2\beta+2\gamma+\delta+\varepsilon+2-2\alpha);$$

$$\alpha < \beta, \quad N_3(p^{2\alpha+1} n) = 0;$$

$$\alpha \leqq \gamma+1, N_3(p^{2\beta+2\alpha} n) = (0; 0, 1, 2\gamma+2-2\alpha, 2\gamma+\delta+2-2\alpha, 2\gamma+\delta+\varepsilon+2-2\alpha);$$

$$N_3(p^{2\beta+2\gamma+2\alpha+2} n) = (2\alpha; 0, 1, 0, \delta, \delta+\varepsilon);$$

$$\alpha \leqq \gamma, \quad N_3(p^{2\beta+2\alpha+1} n) = (0; 1, 0, 2\gamma+1-2\alpha, 2\gamma+\delta+1-2\alpha, 2\gamma+\delta+\varepsilon+1-2\alpha);$$

$$N_3(p^{2\beta+2\gamma+2\alpha+3} n) = (2\alpha+1; 0, 1, 0, \delta, \delta+\varepsilon).$$

The only explicit determinations of (34) other than the classical enumeration of the number of representations of an integer as a sum of five squares appear to be those given in a forthcoming paper.*

For nineteen special enumerations of the type (34) it is there shown that the result (most unexpectedly) can be expressed as a simple function $H(m)$ of the *real divisors alone* of m, when in (34) $p = 2$, $n = m^2$, (an odd square), $a = b = c = d = e = 1$, and three of the numbers β, $\beta+\gamma$, $\beta+\gamma+\delta$, $\beta+\gamma+\delta+\varepsilon$ do not exceed 4, while the fourth is an arbitrary integer > 2. For each of the nineteen forms discussed the enumerations are complete. The number of proper representations is also given. It is easy to verify, however, that not all of the enumerations are independent; some can be derived from others by means of (35) – (37). We shall not stop here to give the complete set of enumerations thus obtainable. The function H is defined by

$$H(m) \equiv \prod[\zeta_3(p^a) - p\,\zeta_3(p^{a-1})],$$

where $m = \prod p^a$ is the prime factor resolution of m, and $\zeta_3(k) =$ the sum of the cubes of all the divisors of k. This function H was introduced by Stieltjes and Hurwitz in determining the number of representations of a square as a sum of five squares. We pass on to the class number relations. The enumerations quoted are proved, in a slightly different notation, in the paper cited.

12. **Class number relations from quinary forms.** Write

$$N(k = x^2 + By^2 + Cz^2 + Du^2 + Ev^2) \equiv N(k),$$

$$N(k = x^2 + By^2) \equiv N_1'(k),$$

$$N(k = C'x^2 + D'u^2 + E'v^2) \equiv N_2'(k),$$

* To be published in the Bulletin of the American Mathematical Society.

where $C = C'G'$, $D = D'G'$, $E = E'G'$, and C', D', E' are co-prime. Then

$$N(k) = \sum_{h=0} N_1'(k - hG')\, N_2'(h), \tag{38}$$

the summation continuing so long as $k - hG' \geqq 0$. There are four similar formulas according as B, C, D or E is chosen in defining N_1'. Similarly if $N(k = B''y^2 + C''z^2) \equiv N_1''(k)$, $N(k = x^2 + Du^2 + Ev^2) \equiv N_2''(k)$, where $B = B''G''$, $C = C''G''$, and B'', C'' are co-prime, we get

$$N(k) = \sum_{h=0} N_2''(h)\, N_1''(k - hG''), \tag{39}$$

one of six like formulas. The method of deriving class number relations by taking N, N_1', N_2' in (38) from the sets of results cited in §§ 11, 5, 8 respectively is obvious, and similarly for (39). A remarkable feature of these relations is that, whereas in general the right of (38) is not a function of the real divisors alone of k, it becomes so when k is an odd square times a power of 2. It will be sufficient to give two of the relations. We found in the paper cited

$$N(4m^2 = x^2 + y^2 + 8z^2 + 8u^2 + 16v^2) = 4H(m),$$
$$N(4m^2 = x^2 + y^2 + 2z^2 + 8u^2 + 16v^2) = 12H(m).$$

In the first, split the form into the two $x^2 + y^2$, $8(z^2 + u^2 + 2z^2)$ and apply (5), (15). The resulting class number relation is

$$12 \sum_{h=0} E(h)\, \xi(m^2 - 2h) = H(m). \tag{40}$$

The left of this is not necessarily an integer multiple of 12, since $E(0) = \frac{1}{12}$.

In the second, split the form into $x^2 + y^2 + z^2$, $8(u^2 + 2v^2)$ and apply (6), (16.2), getting

$$2 \sum_{h=0} \omega(h)\, E(m^2 - 2h) = H(m). \tag{41}$$

If in (40), (41) the odd square m^2 be replaced by n (an arbitrary integer), the sums are not reducible to functions of the real divisors alone of n. These examples sufficiently illustrate the character of the relations which can be found as indicated from the papers quoted.

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NON-MONOIDAL INVOLUTIONS WHICH CONTAIN A WEB OF INVARIANT MONOIDS.

By Virgil Snyder.

1. **Introduction.** In the discussion of rational involutions by means of a (1, 2) correspondence between the points of two spaces made by Sharpe and me* those types were excluded in which the web of invariant surfaces were composed of monoids, since it was supposed they could be more easily obtained by projecting upon a plane from the common vertex. But the same method may be applied to these as to the other cases.

The following paper includes the complete characterization of all the types having a non-composite basis curve of the web of invariant monoids, and a finite number of basis points. That with 7 basis points is monoidal, and is thus included in the category already considered.† Types with 6 points do not exist, while those with 5 or fewer basis points do, thus affording six series of new types of non-monoidal involutions for each integer $n > 2$.

2. **Webs of monoids with a basis curve.** Consider the linear system of surfaces $u_{n-1}x_4 + u_n = 0$, in which u_i is a ternary form of order i in x_1, x_2, x_3. The surfaces of the system have the point $O \equiv (0, 0, 0, 1)$ to multiplicity $n-1$ in common, and have no other basis elements.

Since the surfaces of this system are all rational, the arithmetic genus p_a of each is zero. If within the system a web can be constructed such that any three of its surfaces intersect in two variable points, and the basis elements are independent, it follows from the Riemann-Roch theorem that the variable curve of intersection of any two surfaces of the web has the genus 1. Hence the variable curve lies on a non-singular cubic cone with vertex at O. If the web has a basis curve γ_q of order q, then the variable curve is of order $n^2 - q$, and has O to multiplicity $n^2 - q - 3$. Then the curve γ_q has O for a point of multiplicity $q - 2n + 4$, and has no apparent double points from O. It lies on a non singular cone of order $2(n-2)$, and has the genus $p = (2n-5)(n-3)$. In order for a monoid of the system to contain γ_q, it must have $nq - p + 1$ points on the curve. Of these, $(n-1)(q-2n+4)$ are absorbed at O, hence the monoids through γ_q must have $X = nq - p + 1 - (n-1)(q-2n+4)$ further points in common. In the system of monoids there are $\frac{n(n+1)}{2}$

* Certain types of involutorial space transformations, Trans. Amer. Math. Soc., vol. 20 (1919), pp. 185–202 and vol. 21 (1920), pp. 52–78.

† V. Snyder, Types of monoidal involutions, Ann. of Math., vol. 25 (1924).

$+\frac{(n+1)(n+2)}{2}$ homogeneous constants, and the number of constants in the system through γ_q is $n^2+2n+1-X$.

The residual curve of intersection of any two monoids through γ_q meets γ_q in $6(n-2)$ points apart from O, and it meets an arbitrary surface of the system in $n^2-3n+9-q$ remaining points. If now we impose upon the system of monoids through γ_q the further condition that they shall pass through $n^2-3n+7-q$ simple points P_i not on γ_q, then we have four homogeneous linear constants left. Thus the web determined by γ_q and the points P_i is such that any three of its surfaces intersect in two variable points. Denote a general monoid of this web by ψ_n.

3. **Mapping on a double space.** By associating the surfaces of the web projectively with the planes of another space (x') we have a $(1,2)$ point correspondence between (x') and (x). Since either image in (x) of a given point in (x') uniquely determines the other, these two points are conjugate in an involution I. The order of s, the image of a plane of (x') is n, and the order of s', the image in (x') of a plane of (x), is the same as that of a variable curve of intersection of two monoids of the web, or n^2-q.

$$s_1' \sim s_n\colon\ O^{n-1}\,\gamma_q\colon\ O^{q-2n+4}\ (n^2-3n+7-q)\,P_i$$

$$c_1' \sim c_{n^2-q}\colon\ O^{n^2-q-3},\quad p=1;\quad [c_{n^2-q},\,\gamma_q]=6(n-2),\quad [c_{n^2-q},\,P_i]=1.$$

Since the variable curve c_{n^2-q} is of genus 1, the surface of branch points L' in (x') is of order 4.

Each basis point $P_i \sim \pi_i'$, a plane in (x'); the complete image in (x) of π_i' is P_i and the monoid of the web having a double point at P_i.

The image in (x') of a point of γ_q is a straight line. As the point describes γ_q, the image line describes a ruled surface Γ' of order $6(n-2)$. The jacobian of the web ψ_n is of order $4(n-1)$; it contains O to the multiplicity $(4n-6)$, contains γ_q as a triple curve, and has each P_i as a double point. The cone B_{2n-4} which projects γ_q from O, is composed entirely of parasitic lines; a monoid which passes through an arbitrary point of any one of its generators contains the whole line. Evidently this cone is a component of the jacobian, since each generator meets it in more than $4n-4$ points, the order of the jacobian. The residual is of order $2n$, contains γ_q as a double curve, each P_i as a double point, and has O to the multiplicity $2(n-1)$. Thus a quadratic form in the surfaces of the web. The surface L_4' has for image a surface of order $4n$, but this is taken twice, hence we have $L_4' \sim K_{2n}\colon\ O^{2n-2}\,\gamma_q^2\,P_i^2$. An arbitrary quadric surface in (x') has an image in (x) of the same symbol, but it is self conjugate in I, and its image in (x') is the given quadric surface taken twice.

4. **Fundamental elements.** The image in (x') of B_{2n-4} is a curve of genus $(n-3)(2n-5)$. In I the image of B_{2n-4} is a curve β of order to be determined. A plane s_1 meets B_{2n-4} in a plane curve of order $2n-4$. The image s'_{n^2-q} contains the image curve, hence when the image of s'_{n^2-q} is obtained in (x), this whole cone will come off as a component, hence the order of the involution is $n^3-nq-2n+3$. The basis curve γ_q is of order n^2-q-1 on each surface conjugate to a plane. A plane s_1 will meet its conjugate in a composite curve consisting of c_{2n} with q double points on γ_q and in a residual of order $n^3-nq-4n+3$, having q points of multiplicity n^2-q-3. The points of intersection of these two curves apart from the basis points on γ_q, have for images in (x') the points of contact of the double curve of s'_{n^2-q} with L'. Their number is

$$2n(n^3-nq-4n+3)-2q(n^2-q-3).$$

This is the number of coincidences on the invariant curve s_1, $s_{n^3-nq-4n+3}$. The order of the double curve on s'_{n^2-q} is the number of intersections of this residual curve with a general surface of the web, apart from basis points, divided by 2.

The genus of the residual curve is

$$\frac{(n^3-nq-4n+2)(n^3-nq-4n+1)}{2}-\frac{q}{2}(n^2-q-3)(n^2-q-4).$$

Now by means of Zeuthen's formula we can determine the genus of the double curve of s'_{n^2-q}. If this is denoted by p', we have

$$\begin{aligned}2q(n^2-q-3)-2n(n^3-nq-4n+3)&\\ =4(p'-1)-(n^3-nq-4n+2)(n^3-nq-4n+1)&\\ +q(n^2-q-3)(n^2-q-4)+2.&\end{aligned}$$

The cone B_{2n-4}: $O^{2n-4}\gamma_q$ which projects γ_q from O and K_{2n}: $O^{2n-2}\gamma_q^2(\ )P_i^2$ intersect in γ_q counted as $2q$ and in $4n(n-2)-2q$ generators u, which are fundamental lines of the second kind. These lines lie on the conjugate of every plane in the involution I. The conjugate of a plane meets B_{2n-4} in γ_q counted to multiplicity n^2-q-1 and in the $4n(n-2)-2q$ lines u. The remaining intersection consists of generators of B_{2n-4}; their number is the same as that in which the plane meets the curve β image of B_{2n-4} in I. This curve is therefore of order

$$\begin{aligned}2n^4-4n^3-8n^2+22n-12+q^2-q(3n^2-4n-3)&\\ =2(n-1)(n-2)(n^2+n-3)+q^2-q(3n^2-4n-3).&\end{aligned}$$

5. Table of characteristics. Apart from possible parasitic curves, we now have the basis elements of the involution. Let the surface into which I transforms O be denoted by Ω, and call the image of γ_q Γ. Since we know the order of the jacobian of the web, and the multiplicity of O, γ_q, P_i upon it we may now calculate the orders of Ω and Γ, and the orders of the basis elements upon them.

Apart from fundamental lines of the second kind we may now write the following table:

$s_1 \sim s_{n^2-nq-2n+3}$	: $O^{(n-1)(n^2-q)-2(n-2)}$	$\gamma_q^{n^2-q-1}$	$(n^2-3n+7-q)\,P_i^{n^2-q}\,\beta$
$O \sim \Omega_{n(n^2-q)-5n+4}$	: $O^{(n-1)(n^2-q)-5n+8}$	$\gamma_q^{n^2-q-4}$	$(n^2-3n+7-q)\,P_i^{n^2-q-3}\,\beta$
$\gamma_q \sim \Gamma_{2(3n-1)(n-2)}$	: $O^{2(n-2)(3n-4)}$	$\gamma_q^{6(n-2)}$	$(n^2-3n+7-q)\,P_i^{6(n-2)}\,\beta$
$P_i \sim \psi_{n,i}$	: O^{n-1}	$\gamma_q\,P_i^2$	$(n^2-3n+6-q)\,P$
$\beta \sim B_{2n-4}$	: O^{2n-4}	γ_q.	

The jacobian is of the form

$$J_{4(n^2-nq-2n+2)} \equiv \Omega^2\,\Gamma(n^2-3n+7-q)\,\psi_n^2\,B.$$

6. Fundamental curves of the second kind. The image of a point on such a curve is the whole curve, hence the curve as a whole is transformed into itself. Each such curve appears as a basis curve to multiplicity equal to its order, and on the jacobian to 4 times its order.

We have seen that each generator u common to B_{2n-4} and to K_{2n} is a parasitic line. Similarly, each line ω_i joining P_i to O is a parasitic line; it lies on the ψ_n of the web which has P_i for node.

Pass a plane through OP_1P_2. It meets $\psi_1 = 0$ in ω_1 and in a curve of order $n-1$ which passes through P_1, P_2, the $2(n-2)$ points of γ_q in the plane, and has a point of order $n-2$ at O. The same plane meets $\psi_2 = 0$ in a curve of order $n-1$ through the same points, to the same multiplicity. These curves intersect in O to the order $(n-2)^2$, they have $2(n-2)$ common points on γ_q, and pass through P_1 and P_2. Since

$$(n-2)^2+2(n-2)+1+1 = (n-1)^2+1$$

it follows that the curve is the same on both. Moreover, this curve is a basis curve lying on every

$s_{n^2-nq-2n+3}$,

since

$$(n-1)(n^2-nq-2n+3)-(p=0)+1$$
$$<(n-2)[(n-1)(n^2-q)-2(n-2)]+2(n-2)(n^2-q-1)+2(n^2-q).$$

Therefore among the basis curves of the system, there are as many c_{n-1} as there are planes OP_iP_k, that is

$$\frac{(n^2-3n+7-q)(n^2-3n+6-q)}{2}\,c_{n-1}^{n-1}$$

The complete intersection of ψ_1, ψ_2 is of order n^2. It consists of γ_q, of the plane curve of order $n-1$ just considered, and of a residual of order $n^2-q-n+1$: $O^{n^2-q-n-1}$, $4(n-2)$ pts. on γ_q, and passing simply through each P_i, including P_1 and P_2. It lies on a quadric cone and is therefore rational. Proceeding as before, we see that this curve is also a basis curve on the web of $s_{n(n^2-q)-2n+3}$, since

$$(n^2-nq-2n+3)(n^2-q-n+1)+1$$
$$<(n^2-q-n-1)[(n-1)(n^2-q)-2(n-2)]$$
$$+4(n-2)(n^2-q-1)+(n^2-3n+7-q)(n^2-q).$$

This residual curve belongs to every ψ_i of the system of nodal n-ics of the web of invariant surfaces, since

$$n(n^2-q-n+1)+1$$
$$-\{(n-1)(n^2-q-n-1)+4(n-2)+n^2-q-3n+7\}-1=0.$$

The last term indicates that each ψ_i meets it once more in its node P_i.

Hence an additional basis curve, parasitic for the web $|\psi|$ is

$$c_{n^2-q-n+1},$$

and appears to the multiplicity equal to its order on the web of conjugates of the planes of space.

These two systems of curves completely account for intersections of the nodal surfaces of the web. Hence there can be no other parasitic curves which pass through more than one point P_i, as both the corresponding ψ_i would contain the curve, and that is impossible. For arbitrary positions of γ_q and of P_i the latter cannot lie on curves cutting γ_q in more points than the arbitrary curve of the same system.

6. **Range of value of q.** For any order $n > 3$, the minimum value of q is $n(n-3)$. The curve γ_q lies on a monoid of order $n-3$, which has O for its vertex, and has $(n-3)(n-4)$ generators in common with $B_{2(n-2)}$. There are now 7 basis points P_i. Since the monoid $M \equiv m_{n-3} + m_{n-4} x_4$ and the Geiser net of cubic cones through the seven basis points constitute a new of invariant monoids in the involution, we may write $M\varphi_i = \varrho M' \varphi_i'$, $i = 1, 2, 3$, and for a fourth any surface of the web $\psi_n + x_4 \psi_{n-1} = \varrho(\psi_n' + x_4' \psi_{n-1}')$, indicating with (x') the conjugate of (x) in I. We may now derive the equations of the involution, which is monoidal; $\gamma_{n(n-3)}$ is a simple basis curve. It is included in the general table but contains the extraneous component M. The involution is of order $3(3n-1)$. If we represent the ternary Geiser involution by $\varrho x_i' = G_i(x)$, the equations become

$$\varrho x_i' = G_i[x_4(J_{21}\psi_{n-1} m_{n-4}(G) - \psi_{n-1}(G) m_{n-4}) + J_{21}\psi_n m_{n-4}(G) - \psi_{n-1}(G) m_{n-3}], \quad i = 1, 2, 3;$$

$$\varrho x_4' = -[x_4(J_{21}\psi_{n-1} m_{n-3}(G) - m_{n-4}\psi_n(G)) + J_{21}\psi_n m_{n-3}(G) - \psi_n(G) m_{n-3}],$$

wherein J_{21} is the jacobian of the Geiser net $|G|$ of cones. When $q = n(n-3)+1$, γ_p has O to multiplicity $n^2 - 5n + 5$. Consider the intersection of $B_{2(n-2)}$ and an M_{n-2}. If γ_q is part of the intersection, the residual consists of $n^2 - 5n + 7$ generators of $B_{2(n-2)}$, but this is greater than the whole number of lines on M_{n-2}.

The same argument proves that $\gamma_{n(n-3)+1}$ cannot lie on M_{n-1}. In case of M_n, that is of the same order as the surfaces of the web, the number of generators common to $B_{2(n-2)}$, M_n is $n(n-1)-1$, hence one fewer than the whole number of lines on M_n, but since $2(n-2) = n + n - 1 - 3$, it follows that if $n(n-1)-1$ of the generators common to the two cones of M_n lie on $B_{2(n-2)}$, then the latter contains the other also, hence the residual curve cannot be of order $n(n-3)+1$, but only $n(n-3)$.

If

$$M_n = u_{n-1} x_4 + u_n = 0, \quad B_{2(n-2)} \equiv u_{n-1} v_{n-3} + u_n v_{n-4},$$

and $\gamma_{n(n-3)}$ lies on a monoid of the form $v_{n-4} x_4 - v_{n-3} = 0$, thus presenting the preceding case. Hence we have the theorem:

Involutions with a basis curve of order $n(n-3)+1$, and having 6 simple points do not exist.

If $q = n(n-3)+2$, γ_q lies on an M_{n-2} with all of its lines on $B_{2(n-2)}$. It is the complete intersection of an M_{n-2} and an M_{n-1}.

The equation of the web is therefore $M_{n-2}u_2 + M_{n-1}u_1 = 0$, with five linear conditions satisfied by the five simple basis points. The curve γ, the lines u, the curve β, the five lines ω, the ten plane curves c_{n-1} in the planes OP_iP_k, and the single curve of order $2n-1$ common to all the nodal ψ_n of the web together account for the complete fixed intersection of the web conjugate to the planes of space. The complete characteristic of this case is expressed by

$$s_1 \sim s_{3n^2-4n+3}: O^{3n^2-7n+6}\, \gamma^{3n-3}_{n^2-3n+2}\, 5P_i^{3n-2}$$
$$\beta_{3n^3-10n^2+9n-2}\,(2n^2-2n-4)u\;10\omega\;10c_{n-1}^{n-1}\,c_{2n-1}^{3n-1}.$$

For larger values of q, additional parasitic curves appear. If there are x_i curves of order i, the residual intersection, after accounting for curves analogous to the preceding case, is expressed by the formula $\sum x_i i^3$. Moreover, the complete intersection of s and K_{2n} consists of a plane curve of order $2n$ and of fundamental elements. Since γ is a double curve and all the parasitic curves are simple, we have the totality of the intersection not otherwise accounted for expressed by $\sum x_i i^2$. Since x_i, i are all zero or positive integers, the solution is unique. When $q = n^2-3n+3$, $x_i = 0$ for all i except $i = n-2$, and $x_{n-2} = 4$. Hence there are four curves of order and of multiplicity $n-2$ basis curves of the web $|s|$. One passes through each basis point P.

Similarly, when $q = n^2-3n+4$, we find $x_{n-2} = 0$, $x_{n-3} = 1$, all others zero. Two of the former pass through each basis point. When $q = n^2-3n+5$, $x_{n-2} = 6$, $x_{n-3} = 3$, all others zero. Three curves of order $n-2$ pass through each basis point, but not through both. For $q = n^2-3n+6$ the intersection c_{n-2} no longer appears, as there is but one basis point. We find $x_{n-2} = 4$, $x_{n-3} = 6$, $x_{2n-5} = 1$. The four curves of order $n-2$ pass through the node.

Finally, for $q = n^2-3n+7$, $x_{n-3} = 10$, $x_{2n-6} = 1$, all others zero.

These six forms are not monoidal, and are not reducible to each other. They exist for every value of n, except that for $n = 3$ there is no form without basis points, and that for $n = 3$, $q = 6$ is monoidal, the vertex being the simple basis point. This case, as well as those for $n = 3$, $q = 3$ and $n = 3$, $q = 2$, were already known.*

The other forms, and the complete system of the characteristics, are new. When $n = 3$, $q = 5$, there are two basis points P_1, P_2. The cone B_2

* Clara Moffa, Su alcune corrispondenze birazionali involutorie dello spazio dotate di un sistema lineare di dimensione tre di superficie del terzo ordine unite, Napoli, Tesi di laurea, 1923, 45 pages. See pp. 25–37.

containing γ_5 and any plane of the pencil $P_1 P_2$ is invariant under the involution, hence each plane of the pencil goes into itself. The points of the line $P_1 P_2$ are the images of the generators of the cone B_2. A plane of the pencil meets the cubic surfaces of the web in a net of cubic curves through $P_1 P_2$ and the five points of γ_5 in the plane. Hence the transformation in the plane is the general Geiser involution. The surface K_6 is the locus of the J_6 of the Geiser net in each plane. This transformation of order 9 belongs to a special linear complex.*

Similarly, when $n = 3$, $q = 4$, there are three basis points P_i. The plane of the points P_i is invariant under the involution. The image of B_2 is a plane quartic curve having a node at each P_i. For $n = 4$, $q = 11$, B_4 is a surface of the web of invariant monoids, hence the lines through O correspond to the points of the image plane of B_4 in (x'). All the surfaces of the web $|s_{15}|$ in I have a fixed tangent cone T_{11} at O.

7. **Basis curve composite.** Although the preceding results were obtained upon the assumption that γ_q was non-composite and that B was non-singular, these restrictions are not necessary. If γ_q has a multiple secant from O, each such secant must be a component of it. The tangent cones to the monoids at O may be composite, one component being fixed, or finally, γ_q may have a component in the plane $x_4 = 0$. No one of these cases brings in any essentially new features; the images of the components will together make up Γ of the corresponding general case, hence they are all included in the general classification given above by particularizing the constants, and should not be regarded as new types.

If on the other hand the basis elements are not independent, at least one case appears with quite different characteristics. Consider $n = 3$, $q = 2$. If the basis conic is replaced by two skew lines, one of which passes through O, then in order to have two variable intersections, six basis points may be chosen, and one more comes in. The variable curve of intersection is new of genus 2. This case can be reduced to a sub-case of the involutions discussed by Pieri† when the basis quartic is replaced by a skew quadrilateral. In this form the web of invariant cubics is not composed of monoids; the basiselements include the quadrilateral and four coplanar points.

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* It is a particular case of that given by Sharpe and Snyder, loc. cit., I, p. 195, $n = 0$. The genus 4 is changed to 1 by the presence of the triple point.

† M. Pieri, Sulle tangenti triple di alcune superficie del sest'ordine, Atti di Torino, vol. 24, (1888–9), pp. 514–526.

DEFINITE INTEGRALS CONTAINING A PARAMETER.

By R. L. Jeffery.

The function $f(x, y)$ defined on the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$ is assumed bounded and a Lebesgue integrable function of x for each y; a function $F(y)$ is thus defined by the equation

$$F(y) = \int_a^b f(x, y)\, dx.$$

As was observed by W. H. Young,* the continuity of $F(y)$ follows from the single additional assumption that $f(x, y)$ is continuous in y for each x. The corresponding theorem for Riemann integrals, though the problem naturally received considerable attention, seems not to have been proved until it was made thus obvious through the Lebesgue integral. In the following note the theorem on Riemann integrals is used to establish some facts concerning the uniform convergence of the Riemann sum, $\sum_{i=1}^{n} f(\xi_i, y)\,(x_i - x_{i-1})$, to its limit, $F(y)$.

In line with the conditions under which $F(y)$ was previously known to be continuous we have the following theorem:

Theorem I.† *If the bounded function $f(x, y)$ defined on $a \leqq x \leqq b$, $c \leqq y \leqq d$ be integrable in x for each y and continuous in y uniformly‡ with respect to x, then for a given $\epsilon > 0$ there exists a number $\delta > 0$ such that*

$$\left| F(y) - \sum_{i=1}^{n} f(\xi_i, y)\,(x_i - x_{i-1}) \right| \leqq \epsilon$$

when $x_i - x_{i-1} < \delta$, $(i = 1, 2, \cdots, n)$, independent of y and independent of the choice of ξ_i on the interval (x_{i-1}, x_i).

As we have seen, the conditions on $f(x, y)$ are still sufficient to insure the continuity of $F(y)$ when the assumption that $f(x, y)$ is continuous in y uniformly with respect to x is replaced by the assumption that $f(x, y)$ is continuous in y for each x. The question then naturally arises whether Theorem I remains true when this change is made in the conditions imposed on $f(x, y)$. The following discussion concludes with an example which

* Monatsh. f. Math. u. Phys., vol. 21 (1910), pp. 126–127.

† Gillespie, Bull. Amer. Math. Soc., ser. 2, vol. 18 (1912), p. 381. It was at Prof. Gillespie's suggestion that I undertook a further study of this problem.

‡ This implies that for a given $\epsilon > 0$ there corresponds to each value y_0 of y a number $\delta > 0$ and independent of x such that $|f(x, y_0) - f(x, y)| \leqq \epsilon$ when $|y_0 - y| < \delta$.

shows that this is not the case. However, even when the condition of uniformity is yielded, it is still possible to obtain a theorem very similar to Theorem I, the difference being that the choice of ξ_i on the interval (x_{i-1}, x_i) is not arbitrary.

THEOREM II. *If the bounded function $f(x, y)$ defined on $a \leqq x \leqq b$, $c \leqq y \leqq d$ be Riemann integrable in x for each y and continuous in y for each x, then for a given $\varepsilon > 0$ there exists a number $\delta > 0$, and a certain set of x-points G_ε, such that*

$$\left| F(y) - \sum_{i=1}^{n} f(\xi_i, y)\,(x_i - x_{i-1}) \right| \leqq \varepsilon$$

when $x_i - x_{i-1} < \delta$ and ξ_i is any point of G_ε on the interval (x_{i-1}, x_i) if there are points of G_ε on this interval; otherwise ξ_i is any point of the interval (x_{i-1}, x_i).

The pairs of rational values (y'_r, y''_r) of y on the interval (c, d) which satisfy the inequality $|y'_r - y''_r| < \delta_1 > 0$ form a countable set. We designate such pairs in an ordered array by (y'_{r_1}, y''_{r_1}), (y'_{r_2}, y''_{r_2}), $\cdots$, $(y'_{r_i}, y''_{r_i}) \cdots$. The function $f(x, y)$ being Riemann integrable in x for each y, the set G_{r_i} of x-points at which

$$|f(x, y'_{r_i}) - f(x, y''_{r_i})| \leqq \frac{\varepsilon}{8(b-a)}$$

is measurable; hence the set $G_1 \equiv D(G_{r_1}, G_{r_2} \cdots, G_{r_i} \cdots)$, of points common to all the sets of the sequence, is measurable.* The set G_1 is now the set of x-points at which

$$|f(x, y'_r) - f(x, y''_r)| \leqq \frac{\varepsilon}{8(b-a)}$$

for every pair of rational values of y satisfying the inequality $|y'_r - y''_r| < \delta_1$. Furthermore, since $f(x, y)$ is continuous in y for each x, it follows that G_1 is the set of x-points at which

$$|f(x, y') - f(x, y'')| \leqq \frac{\varepsilon}{8(b-a)}$$

for every pair of values of y, rational or irrational, which satisfy the inequality $|y' - y''| < \delta_1$. If now we make δ_1 the first number of a monotonically decreasing sequence, $\delta_1, \delta_2, \cdots, \delta_i, \cdots$, which approaches zero as a limit, we can use this sequence to define an infinite sequence of measurable sets, $G_1, G_2, \cdots, G_i, \cdots$, where G_i is the set of x-points at which

$$|f(x, y') - f(x, y'')| \leqq \frac{\varepsilon}{8(b-a)}$$

* Hobson, Functions of a real variable, 2nd ed., vol. 1, § 131.

when y' and y'' are any two values of y on the interval (c, d) which satisfy the inequality $|y'-y''|<\delta_i$.

All the sets $G_1, G_2, \cdots, G_i$ being measurable, the set $G \equiv M(G_1, G_2, \cdots, G_i)$, which is the set of all points belonging to any G_i is measurable.* Furthermore, since $f(x, y)$ is a continuous function of y for each x, the set G contains all the points of the interval (a, b); and since G_i contains G_{i-1}, the limit, as i becomes infinite, of the measure of G_i is equal to the measure of G;† or in symbols $\lim_{i\to\infty} mG_i = mG = (b-a)$. We denote now by M and m the least upper bound and the greatest lower bound respectively of $f(x, y)$ on the fundamental rectangle; choose a value k of i sufficiently large to insure that $mG_k > (b-a) - \varepsilon/8(M-m)$ and prove that this set G_k is the set G_ε announced in the theorem. Over this set then, which we shall designate by G_ε, we have

$$(1) \qquad |f(x, y') - f(x, y'')| \leqq \frac{\varepsilon}{8(b-a)}$$

for all pairs of values y' and y'' of y on the interval (c, d) which satisfy the inequality $|y'-y''|<\delta_k$ and

$$(2) \qquad mG_\varepsilon > (b-a) - \frac{\varepsilon}{8(M-m)}$$

Let us now choose y_0, any value of y on the interval (c, d) and a number $\eta_0 > 0$, such that if we divide the interval $a \leqq x \leqq b$ into sub-intervals (x_{i-1}, x_i), $(i = 1, 2, \cdots, n)$ and denote by $M_i(y_0)$ and $m_i(y_0)$ the least upper bound and greatest lower bound, respectively, of $f(x, y_0)$ on the interval (x_{i-1}, x_i) we shall have

$$(3) \qquad \left| F(y_0) - \sum_{i=1}^{n} M_i(y_0)\,(x_i - x_{i-1}) \right| \leqq \frac{\varepsilon}{2},$$

and

$$(4) \qquad \left| F(y_0) - \sum_{i=1}^{n} m_i(y_0)\,(x_i - x_{i-1}) \right| \leqq \frac{\varepsilon}{2},$$

provided $x_i - x_{i-1} < \eta_0$. Having chosen any subdivision for which (3) and (4) both hold, let us denote the part of G_ε on the interval (x_{i-1}, x_i) by E_i, and the part of $C(G_\varepsilon)$, the set complementary to G_ε, on this interval, by e_i, and the measure of these two sets by $\overline{E}_i$ and $\overline{e}_i$ respectively. Now let

* Hobson, 2nd. ed., vol. 1, § 130.

† Hobson, 2nd ed., vol. 1, § 131.

$$S_1 = F(y_0) - \sum_{i=1}^{n} f(x_i^E, y_0)\,\bar{E}_i - \sum_{i=1}^{n} f(x_i^e, y_0)\,\bar{e}_i$$

where x_i^E and x_i^e are any points of E_i and e_i respectively. If an interval (x_{i-1}, x_i) is made up exclusively of points of e_i, the term of S_1 denoted by $f(x_i^E, y_0)\,\bar{E}_i$ is zero; a similar remark applies to an interval containing only points of E_i. Obviously since

$$m_i(y_0) \leqq f(x_i^E, y_0) \leqq M(y_0), \quad \text{and} \quad m_i(y_0) \leqq f(x_i^e, y_0) \leqq M_i(y_0),$$

we have

$$|S_1| \leqq \frac{\varepsilon}{2}. \tag{5}$$

Let us now define the function

$$Q(y) \equiv F(y) - \sum_{i=1}^{n} f(x_i^E, y)\,\bar{E}_i - \sum_{i=1}^{n} f(x_i^e, y)\,\bar{e}_i$$

and determine an interval about y_0 for which $|Q(y)| < 7\varepsilon/8$.

If

$$S_2 = F(y) - F(y_0),$$

$$S_3 = \sum_{i=1}^{n} f(x_i^E, y_0)\,\bar{E}_i - \sum_{i=1}^{n} f(x_i^E, y)\,\bar{E}_i,$$

$$S_4 = \sum_{i=1}^{n} f(x_i^e, y_0)\,\bar{e}_i - \sum_{i=1}^{n} f(x_i^e, y)\,\bar{e}_i,$$

then

$$Q(y) \equiv S_1 + S_2 + S_3 + S_4.$$

Hence

$$|Q(y)| \leqq |S_1| + |S_2| + |S_3| + |S_4|.$$

But from (5), $|S_1| < \varepsilon/2$; and since $F(y)$ is continuous at y_0, there exists a number $\gamma > 0$ such that $|S_2| \equiv |F(y) - F(y_0)| < \varepsilon/8$ when $|y - y_0| < \gamma$. Furthermore, it follows from (1) that $|S_3| < (b-a)\dfrac{\varepsilon}{8(b-a)} = \dfrac{\varepsilon}{8}$ for $|y - y_0| < \delta_k$, and from (2) that $|S_4| < (M-m)\dfrac{\varepsilon}{8(M-m)} = \dfrac{\varepsilon}{8}$ for every y on the interval (c, d). Hence for γ_0 less than the smaller of the two numbers γ and δ_k, we see that

$$\left| F(y) - \sum_{i=1}^{n} f(x_i^E, y)\,\bar{E}_i - \sum_{i=1}^{n} f(x_i^e, y)\,\bar{e}_i \right| < \frac{7\varepsilon}{8}, \tag{6}$$

provided $|y - y_0| < \gamma_0$.

Let us consider the change in the left hand side of this inequality, if in place of $\sum_{i=1}^{n} f(x_i^e, y)\,\bar{e}_i$ we substitute $\sum_{i=1}^{n} f(x_i^E, y)\,\bar{e}_i$. Writing the absolute value of the difference of these two sums, it is seen from (2) that

$$\left|\sum_{i=1}^{n}\left\{f(x_i^e, y) - f(x_i^E, y)\right\}\bar{e}_i\right| < \frac{\varepsilon}{8} \tag{7}$$

Hence $\left|F(y) - \sum_{i=1}^{n} f(x_i^E, y)\,\bar{E}_i - \sum_{i=1}^{n} f(x_i^E, y)\,\bar{e}_i\right| < \frac{7\varepsilon}{8} + \frac{\varepsilon}{8}$, or

$$\left|F(y) - \sum_{i=1}^{n} f(x_i^E, y)\,(x_i - x_{i-1})\right| < \varepsilon \tag{8}$$

for all values of y on the interval $(y_0 - \gamma, y_0 + \gamma)$, provided $x_i - x_{i-1} < \eta_0$. The last inequality may be written*

$$\left|F(y) - \sum_{i=1}^{n} f(\xi_i, y)\,(x_i - x_{i-1})\right| < \varepsilon \tag{9}$$

for all values of y on the interval $(y_0 - \gamma_0, y_0 + \gamma_0)$, provided $x_i - x_{i-1} < \eta_0$ and ξ_i is any point of G_ε on the interval (x_{i-1}, x_i) if this interval contains points of G_ε; otherwise ξ_i is any point of the interval (x_{i-1}, x_i).

The foregoing reasoning shows that to each value y_0 of y on the interval (c, d) there corresponds a number $\eta_0 > 0$, and a number $\gamma_0 > 0$, such that (8) holds for all values of y on the interval $(y_0 - \gamma_0, y_0 + \gamma_0)$ whenever $x_i - x_{i-1} < \eta_0$. Thus all the points of the interval $c \leqq y \leqq d$ lie on an infinite set of overlapping intervals, and therefore, by virtue of the Heine-Borel theorem, lie on a properly selected finite number of intervals of this set.

To each interval of this finite set there corresponds a value of $\eta_0 > 0$ such that (8) holds for all values of y on the interval, provided $x_i - x_{i-1} < \eta_0$. Hence if δ is chosen smaller than the smallest of this finite number of values represented by η_0, we shall have $\left|F(y) - \sum_{i=1}^{n} f(\xi_i, y)\,(x_i - x_{i-1})\right| < \varepsilon$ for all values of y on the interval (c, d), provided $x_i - x_{i-1} < \delta$, and ξ_i is any point of G_ε on the interval (x_{i-1}, x_i) if there are points of G_ε on this interval; otherwise ξ_i is any point of the interval (x_{i-1}, x_i).

* This becomes evident by considering that for any interval, (x_{i-1}, x_i), which is made up exclusively of points of $C(G_\varepsilon)$, the part of (6) denoted by $f(x_i^E, y)\,\bar{E}_i$ is zero, and the part denoted by $f(x_i^e, y)\,\bar{e}_i$ becomes $f(\xi_i, y)\,(x_i - x_{i-1})$; hence in (7) we need consider only the values of i for which the corresponding intervals (x_{i-1}, x_i) contain points of both G_ε and $C(G_\varepsilon)$.

THEOREM III. *If the bounded function $f(x, y)$ is monotone in x for each y and continuous in y for each x, then for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\left| F(y) - \sum_{i=1}^{n} f(\xi_i, y)\,(x_i - x_{i-1}) \right| < \varepsilon$$

when $x_i - x_{i-1} < \delta$ and ξ_i is any point on the interval (x_{i-1}, x_i).

Let M and m be the upper and lower bounds respectively of $f(x, y)$, choose $\delta < \varepsilon/2(M-m)$, and at the same time sufficiently small to insure that

$$\left| F(y) - \sum_{i=1}^{n} f(\xi_i, y)\,(x_i - x_{i-1}) \right| < \frac{\varepsilon}{2} \tag{1}$$

when $x_i - x_{i-1} < \delta$ and ξ_i is chosen in accordance with Theorem II. Now consider any subdivision whatever of (a, b) such that $x_i - x_{i-1} < \delta$. Suppose it is possible to determine a value y_ε of y, and values ξ_i^ε of x on the subdivisions (x_{i-1}, x_i) for which the inequality

$$\left| F(y_\varepsilon) - \sum_{i=1}^{n} f(\xi_i^\varepsilon, y_\varepsilon)\,x_i - x_{i-1}) \right| > \varepsilon \tag{2}$$

holds. Fixing $y = y_\varepsilon$ in (1) and combining with (2) we get

$$\left| \sum_{i=1}^{n} f(\xi_i, y_\varepsilon)\,(x_i - x_{i-1}) - \sum_{i=1}^{n} f(\xi_i^\varepsilon, y_\varepsilon)\,(x_i - x_{i-1}) \right| > \frac{\varepsilon}{2}$$

$$\left| \sum_{i=1}^{n} (x_i - x_{i-1})\,\{f(\xi_i, y_\varepsilon) - f(\xi_i^\varepsilon, y_\varepsilon)\} \right| > \frac{\varepsilon}{2} \tag{3}$$

$$\frac{\varepsilon}{2(M-m)} \sum_{i=1}^{n} |f(\xi_i, y_\varepsilon) - f(\xi_i^\varepsilon, y_\varepsilon)| > \frac{\varepsilon}{2},$$

hence, since $f(x, y_\varepsilon)$ is monotone in x,

$$\frac{\varepsilon}{2(M-m)}\,(M-m) > \frac{\varepsilon}{2}$$

$$\frac{\varepsilon}{2} > \frac{\varepsilon}{2}.$$

Hence (2) cannot exist.

In conclusion we define on the rectangle $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$ *a function $f(x, y)$ which is bounded and continuous in each variable separately, but which, nevertheless, has the following property—a number $\varepsilon > 0$ being*

given, there will exist for any subdivision whatever of the interval $0 \leqq x \leqq 1$ *a value* y_ε *of* y, *and on each subdivision* (x_{i-1}, x_i) *a value* ξ_i *of* x *such that*

$$\left| F(y_\varepsilon) - \sum_{i=1}^{n} f(\xi_i, y_\varepsilon)(x_i - x_{i-1}) \right| > \frac{1}{2} - \varepsilon.$$

A function $Q(x, y)$ will first be defined on the rectangle $A \equiv (1, 0)$, $B \equiv (1, 1)$, $C \equiv (-1, 1)$, $D \equiv (-1, 0)$ as follows: Draw the line joining the points A and $E \equiv (0, 1)$. The equation of this line is $y = 1 - x$.

$Q(x, y) = xy/(1-x)$ at all points of the rectangle, *ABEO* (where O is the origin) below or on the line $y = 1 - x$ except the point $(1, 0)$;

$Q(x, y) = (1-x)(1-y)/y$ at all points of the rectangle *ABEO* above the line $y = 1 - x$;

$Q(1, 0) = 0$;

$Q(x, y)$ is defined on the rectangle *CDOE* by a reflection with respect to the y-axis.

It is easily seen that $Q(x, y)$ is bounded and a continuous function of each variable separately; it is a continuous function of the two variables except at the points $(1, 0)$ and $(-1, 0)$, where it has a saltus equal to unity; it is zero on the bounding lines of the rectangle.

We shall now define on the interval $0 \leqq x \leqq 1$ a set of open intervals, $\beta_1, \beta_2, \cdots, \beta_i, \cdots$, as follows: divide the interval $(0, 1)$ into thirds and let β_1 be the interior points of the middle third; bisect each of the remaining thirds, and about the points of bisection as middle points, construct intervals each of length $1/3^2 \cdot 2$; let β_2 be the interior points of these two intervals. In each of the remaining four intervals, construct a similarly situated interval of length $1/3^3 2^2$; let β_3 be the interior points of these four intervals. The law of formation of the set, $\beta_1, \beta_2, \cdots, \beta_i, \cdots$, is apparent, and obviously the measure of $\beta \equiv M(\beta_1, \beta_2, \cdots, \beta_i, \cdots)$ is $1/3 + 1/3^2 + 1/3^3 + \cdots$ $\cdots = 1/2$. Hence the content of the Cantor non-dense perfect set ω complementary to the set β is equal to $1/2$.

Obviously every point of the perfect set ω is a limiting point of the end points of the set of open intervals, β; and, since the content of a set of points is the same as the content of its derivative,* it follows that the content of the end points of β-intervals is $1/2$.

On the interval β_1 construct a rectangle $A'B'C'D'$, whose width is one half of the interval, β_1. Between this rectangle and the rectangle, *ABCD*, on which $Q(x, y)$ is defined, set up a one to one correspondence by a projective transformation which carries A into A', B into B', C into C', and

* Hobson, 2nd ed., vol. 1, p. 155.

D into D'. At each point of the rectangle, $A'B'C'D'$, let $f(x, y)$ have the value of $Q(x, y)$ at the corresponding point on the rectangle, $ABCD$; repeat this process for each interval of the set, $\beta_1, \beta_2, \cdots, \beta_i, \cdots$. At all points of the rectangle, $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, exterior to the set of rectangles defined on the intervals $\beta_1, \beta_2, \cdots, \beta_i, \cdots$, let $f(x, y) = 0$.

The function $f(x, y)$ is bounded, continuous in x for each y, and continuous in y for each x; hence $F(y) = \int_0^1 f(x, y)\,dx$ is a continuous* function of y. Now let a number $\varepsilon > 0$ be given, and consider, $0, x_1, x_2, \cdots, x_{n-1}, 1$, any subdivision whatever of the interval, $(0, 1)$. Suppose there are p of these n sub-divisions which contain end points of the β-intervals. The sum of the lengths of these p sub-divisions is $\geqq 1/2$. The function $F(y)$ is continuous and $F(0) = 0$; hence there exists a number $\delta > 0$ such that

$$F(y) < \frac{\varepsilon}{2} \text{ for } y < \delta. \tag{1}$$

If for any i, the interval, (x_{i-1}, x_i), is one of the p sub-divisions which contain end points of the β-intervals, there corresponds to this sub-division a value y_p of y, such that for each $y \leqq y_p$, a value ξ_i of x on the interval, (x_{i-1}, x_i), can be found for which†

$$f(y, \xi_i) > 1 - \varepsilon. \tag{2}$$

If now $y_\varepsilon > 0$ is chosen smaller than the smallest of the $p+1$ numbers represented by δ and y_p, and if ξ_i is chosen in accordance with (2) for each of the p sub-divisions containing end points of the β-intervals, and for the remaining $n - p$ sub-divisions is any point of the sub-divisions, we get

$$\left| F(y_\varepsilon) - \sum_{i=1}^{n} f(\xi_i, y_\varepsilon)(x_i - x_{i-1}) \right| \geqq \left| \frac{\varepsilon}{2} - (1-\varepsilon)\frac{1}{2} \right| = \frac{1}{2} - \varepsilon.$$

In the foregoing ε is arbitrary; we can conclude, therefore, that Theorem I is no longer true when the assumption that $f(x, y)$ is continuous in y uniformly with respect to x is replaced by the assumption that $f(x, y)$ is continuous in y for each x.

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* W. H. Young; loc. cit.

† This becomes evident when we remember that for each end point $(x_\beta, 0)$ of the β-intervals there is a line along which $f(x, y)$ approaches unity as the point (x, y) approaches the point $(x_\beta, 0)$, and at the same time $f(x_\beta, 0) = 0$.

INTEGRO-DIFFERENTIAL EXPRESSIONS INVARIANT UNDER VOLTERRA'S GROUP OF TRANSFORMATIONS.*

By Aristotle D. Michal.

The study of continuous groups of transformations in function space was begun by G. Kowalewski† in 1911. Vessiot‡ considered the conditions under which r-parameter Volterra transformations form a group. L. L. Dines§ considered projective transformations in function space continuing Kowalewski's work. I. A. Barnett|| gave a few examples of functionals invariant under one-parameter continuous groups of transformations in the space of continuous functions.

The systematic study of functionals invariant under continuous groups of transformations in function space, and also the study of functionals involving derivatives of functions and invariant under groups of transformations in function space have not been considered previously, so far as the writer has been able to ascertain. Consequently the author commences the study of the latter functionals, i. e., functionals involving derivatives of functions (and as a special case, integro-differential equations), that are invariant under a Volterra one-parameter group of continuous transformations, a problem allied to the calculus of variations.

In Part I of this paper I consider a sufficient condition for the invariance of such functionals. In Part II, necessary and sufficient conditions for invariance are discussed and the actual calculation of a large class of such functionals, known as functionals developable in a Volterra expansion ("analytic functionals"), is effected.

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* Preliminary Report to the American Mathematical Society, April, 1923.

† G. Kowalewski, Sitzb. Math. Akad. Wiss. Wien, 1911.

‡ E. Vessiot, Comptes Rendus, t. 154 (1912), pp. 57 and 682.

§ L. L. Dines, Trans. of the Amer. Math. Soc., vol. 20 (1919), and Amer. Jour. of Math. April, 1922.

|| I. A. Barnett, Proc. Nat. Acad. Sci. U. S. A., vol. 6 (1920).

PART I.

1. **Statement of problem.** Consider the problem of finding the functionals of the form $f[y(\overset{1}{\underset{0}{\tau}}), y'(\overset{1}{\underset{0}{\tau}})]$ that are invariant under a Volterra group of transformations

$$y_1(x) = y(x) + \int_0^x K(x, s/a)\, y(s)\, ds, \tag{1}$$

where a is the parameter of this continuous one-parameter group of transformations; $y'(\tau)$ denotes the derivative of $y(\tau)$ with respect to τ; $f[y(\overset{1}{\underset{0}{\tau}}), y'(\overset{1}{\underset{0}{\tau}})]$ depends on $y(\tau)$ and $y'(\tau)$ only between 0 and 1. *We restrict ourselves to transformations* (1) *for which the* $y'(\tau)$*'s exists and are continuous. We further restrict ourselves to transformations* (1) *for which* $K(x, s/a)$ *is continuous in* x *and* s; $\dfrac{\partial K(x, s/a)}{\partial x}$ *is continuous in* x *and* s; *and* $K(s, s/a) \neq 0$, *the value of* a *not being that which corresponds to the identical transformation.* The infinitesimal transformation of (1) will be of the form

$$\delta y(x) = \left[\int_0^x H(x, s)\, y(s)\, ds\right] \delta a. \tag{2}$$

If $f[y(\overset{1}{\underset{0}{\tau}}), y'(\overset{1}{\underset{0}{\tau}})]$ is invariant under (1) for all y's and their corresponding first derivatives, then it will follow, in particular, that the integro-differential equation

$$f[y(\overset{1}{\underset{0}{\tau}}), y'(\overset{1}{\underset{0}{\tau}})] = 0$$

is invariant under such transformations.

2. **Sufficient condition for invariance.** THEOREM 1. *If* $f[y(\overset{1}{\underset{0}{\tau}}), y'(\overset{1}{\underset{0}{\tau}})]$ *has a variation of Volterra form**

$$\delta f[y(\overset{1}{\underset{0}{\tau}}), y'(\overset{1}{\underset{0}{\tau}})] = \int_0^1 f_y[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}/r]\, \delta y(r)\, dr + \int_0^1 f_{y'}[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}/r]\, \delta y'(r)\, dr$$

in which $f_y[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}/r]$ *and* $f_{y'}[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}/r]$, *respectively, stand for the partial functional derivatives of* $f[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}]$ *with respect to* y *and* y' *taken at the point* r *and are continuous as functions of* r, *it is sufficient in order that* $f[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}]$ *be invariant of a group of Volterra transformations* (1) *that it satisfy the following equation in partial functional derivatives*:

* A more general expression for δf would be in the form of Stieltjes integrals.

$$f_{y'}[\underset{0}{\overset{1}{y}}, \underset{0}{\overset{1}{y'}}/s] = \int_0^1 L(s,r)\, f_y[\underset{0}{\overset{1}{y}}, \underset{0}{\overset{1}{y'}}/r]\, dr,$$

where

$$L(s,r) = \int_s^r k(s,r')\frac{H(r,r')}{H(r',r')}\,dr' - \frac{H(r,s)}{H(s,s)} \text{ when } r \geqq s,$$
$$= 0 \text{ when } r < s;$$

and $k(s,r')$ is the reciprocal kernel of

$$\frac{-1}{H(s,s)}\frac{\partial H(r',s)}{\partial r'}.$$

Proof. A necessary and sufficient condition that $f[y(\underset{0}{\overset{1}{\tau}}), y'(\underset{0}{\overset{1}{\tau}})]$ be invariant under (1) is that under (2)

$$\delta f[y(\underset{0}{\overset{1}{\tau}}), y'(\underset{0}{\overset{1}{\tau}})] \equiv 0 \text{ in } y \text{ and } y'. \tag{3}$$

Using Volterra's form of the variation of a functional, condition (3) becomes

$$\delta f[y(\underset{0}{\overset{1}{\tau}}), y'(\underset{0}{\overset{1}{\tau}})] = \int_0^1 f_y[\underset{0}{\overset{1}{y}}, \underset{0}{\overset{1}{y'}}/r]\,\delta y(r)\,dr + \int_0^1 f_{y'}[\underset{0}{\overset{1}{y}}, \underset{0}{\overset{1}{y'}}/r]\,\delta y'(r)\,dr = 0. \tag{4}$$

Evidently we have

$$\delta y'(r) = \frac{d[\delta y(r)]}{dr} = \left[H(r,r)\,y(r) + \int_0^r H_1(r,s)\,y(s)\,ds\right]\delta a, \tag{5}$$

$H_1(r,s)$ standing for the first partial derivative of $H(r,s)$ with respect to r.

Substituting in (4) the values of $\delta y(r)$ as given by (2) and the values of $\delta y'(r)$ as given by (5), we get, on dividing through by δa,

$$\int_0^1 f_y[\underset{0}{\overset{1}{y}}, \underset{0}{\overset{1}{y'}}/r]\,dr \int_0^r H(r,s)\,y(s)\,ds + \int_0^1 f_{y'}[\underset{0}{\overset{1}{y}}, \underset{0}{\overset{1}{y'}}/r]\,H(r,r)\,y(r)\,dr$$
$$+ \int_0^1 f_{y'}[\underset{0}{\overset{1}{y}}, \underset{0}{\overset{1}{y'}}/r]\,dr \int_0^r H_1(r,s)\,y(s)\,ds = 0 \quad (0 \leqq s \leqq 1).$$

Changing r to s in the single integral term and using Dirichlet's formula, we get

$$(6)\qquad \int_0^1 y(s)\,ds\Big[\int_s^1 \{f_y[\overset{1}{y}_0, \overset{1}{y}_0'/r]H(r,s)+f_{y'}[\overset{1}{y}_0, \overset{1}{y}_0'/r]H_1(r,s)\}\,dr + f_{y'}[\overset{1}{y}_0, \overset{1}{y}_0'/s]H(s,s)\Big] = 0$$

for all y.

The results of the following lemma will be of use:

LEMMA. *Necessary and sufficient conditions that* $H(x,s)$, $\dfrac{\partial H(x,s)}{\partial x}$ *be continuous in* x *and* s *and that* $H(s,s) \neq 0$ *are that* $K(x, s/a)$, $\dfrac{\partial K(x,s/a)}{\partial x}$ *be continuous in* x *and* s *and that* $K(s, s/a) \neq 0$, *where* a *does not correspond to the identical transformation.*

Proof. The proof of this lemma comes from considering the expressions giving the relation of $K(x, s/a)$ and $H(x, s)$. It is known* that

$$1+jK(x,s/a) = e^{ajH(x,a)},$$

where $a = 0$ corresponds to the identical transformation. Thus

$$\log(1+jK) = jHa = jK-\frac{jK^2}{2}+\frac{jK^3}{3}-\cdots$$

or

$$(7)\qquad H(x,s) = \Big[\frac{\partial K(x,s/a)}{\partial a}\Big]_{a=0} = \frac{1}{a}\Big(K-\frac{1}{2}K^2+\frac{1}{3}K^3-\cdots\Big),$$

the exponentials denoting symbolic multiplication according to the convention of Volterra.

By direct calculation from series (7), the sufficiency of the condition is established. On the other hand, from the series

$$(8)\qquad K(x,s/a) = aH+\frac{a^2H^2}{2!}+\frac{a^3H^3}{3!}+\cdots$$

the necessity of the condition is established by direct calculation.

In order that (6), an identity in y, be satisfied, it is *sufficient* that the expression in brackets be equal to zero. Rearranging and dividing by $H(s,s)$ (since $H(s,s)$ does not vanish), we get

$$(9)\qquad f_{y'}[\overset{1}{y}_0, \overset{1}{y}_0'/s] = -\int_s^1 f_y[\overset{1}{y}_0, \overset{1}{y}_0'/r]\frac{H(r,s)}{H(s,s)}\,dr - \int_s^1 f_{y'}[\overset{1}{y}_0, \overset{1}{y}_0'/r]\frac{H_1(r,s)}{H(s,s)}\,dr \qquad (0 \leqq s \leqq 1).$$

* G. Kowalewski, Comptes Rendus, vol. 153 (1911). For the notation and the complex algebra in use, see G. C. Evans' "Cambridge Colloquium Lectures," p. 119. The rigorous justification of the relation is very simple in terms of this notation.

Equation (9) is a Volterra integral equation of the second kind; i. e., it is of the form

$$f(s) = \varphi(s) + \int_s^1 G(s, r) f(r)\, dr,$$

where the kernel $G(s, r) = -\dfrac{H_1(r, s)}{H(s, s)}$ is continuous. *Such an equation has one and only one continuous solution.* This solution is given by

$$f_{y'}[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}/s] = \psi[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}/s] - \int_s^1 k(s, r)\, \psi[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}/r]\, dr, \tag{10}$$

where

$$\psi[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}/s] = -\int_s^1 f_y[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}/r] \frac{H(r, s)}{H(s, s)}\, dr$$

and where $k(s, r)$ is the reciprocal kernel of $-\dfrac{H_1(r, s)}{H(s, s)}$.

We can transform (10) into the form

$$f_{y'}[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}/s] = \int_0^1 L(s, r)\, f_y[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}/r]\, dr, \tag{11}$$

where

$$L(s, r) = \int_s^r k(s, r') \frac{H(r, r')}{H(r', r')}\, dr' - \frac{H(r, s)}{H(s, s)}, \qquad r \geqq s;$$
$$= 0, \qquad r < s.$$

Thus Theorem 1 has been proved.

3. Integrability conditions for functional equation (11). Equation (11) is an equation in partial functional derivatives. We shall have to investigate the conditions under which (11) is integrable.* Lévy's analysis is applicable to equation (11).

In the course of the discussion of the integrability conditions of functional equation (11), we shall prove the following theorem:

THEOREM 2. *There is† one and only one solution $f[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}]$ of* (11), *regarded momentarily as a functional of two independent arguments, such that for a given initial functional value of the second argument*

$$y'(\tau) = y_0'(\tau)$$

$f[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}]$ *has arbitrary values as a functional of its first argument.*

* P. Lévy, Analyse fonctionnelle, 2nd part, Chapters 1 & 4.

† It is evident from the calculations of Part II that solutions of (11) exist.

Proof. Let us investigate the integrability conditions of an equation of the form

$$u_z = \varphi\,[\overset{1}{\underset{0}{u_y}}], \tag{12}$$

where u is a functional of the functions y and z and u_y, u_z stand respectively for the partial functional derivatives of u with respect to y and z.

The notion of an adjoint functional is necessary for the condition of integrability. So we lay down Lévy's definition and say that a functional E is adjoint to the functional $\mathfrak{E}$ if

$$\int_0^1 y(t)\,E[z]\,dt = \int_0^1 z(t)\,\mathfrak{E}[y]\,dt$$

for all continuous functions $y(t)$ and $z(t)$.

The condition that (12) be integrable is, according to Lévy,* that

$$\delta u_y = E[\delta y] + F[\delta z], \qquad \delta u_z = \mathfrak{F}[\delta y] + G[\delta z], \tag{13}$$

where E and G are self-adjoint linear functionals, and F and $\mathfrak{F}$ are functionals adjoint to each other.

Here follows a more explicit statement of the integrability condition for (12): Assume provisionally that (12) has been solved, $z(\tau)$ ranging according to a definite law. Then suppose (12) has been integrated to a function $z = Z$. The condition that the solution $u[y, z]$ of (12) be independent of the intermediate values of z between the initial given value $z = z_0$ and $z = Z$ constitutes the condition of integrability for the functional equation (12).

Let us impose the Lévy conditions (13) in our case and see what the integrability condition is. Given that the equation (12) has been integrated as far as a certain functional value $z = \bar{z}$, u becomes a given functional of y. Thus u_y and $E[\delta y]$ are given functionals and the condition that $E[\delta y]$ is self-adjoint is therefore verified.

From (12) we see that

$$\delta u_z = H[\delta u_y], \tag{14}$$

where H is a linear functional in δu_y. Replacing δu_y and δu_z in (14) by (13), we get

$$\begin{aligned}\mathfrak{F}[\delta y] + G[\delta z] &= H[E[\delta y] + F[\delta z]] \\ &= H[E[\delta y]] + H[F[\delta z]].\end{aligned}$$

* P. Lévy, Analyse fonctionnelle, pp. 98, 212.

Equating terms in δy and δz separately, we get

$$\mathfrak{F}[\delta y] = H[E[\delta y]], \qquad G[\delta z] = H[F[\delta z]]. \tag{15}$$

Imposing the condition that $\mathfrak{F}$ be adjoint to F, we have, by the first equation of (15),

$$F[\delta z] = E[\mathfrak{H}[\delta z]], \tag{16}$$

where $\mathfrak{H}$ is the adjoint functional of H. Putting (16) in the second equation of (15), we get

$$G[\delta z] = H\{E[\mathfrak{H}[\delta z]]\}. \tag{17}$$

Since $H[E]$ is the adjoint functional of $E[\mathfrak{H}]$, it is seen, as Lévy points out, that any functional of the form (17) is self-adjoint automatically. In our case, then, the integrability conditions are satisfied identically and consequently (12) is completely integrable. Hence given u for some initial function $z(\tau) = z_0(\tau)$ arbitrarily as a functional of y, there is one and only one solution $u[y, z]$ which takes on these initial values, and is defined for all functions $z(\tau)$ in the neighborhood of $z_0(\tau)$ and all functions $y(\tau)$.

In particular, if $z(\tau) \equiv y'(\tau)$, a special case of (12) will become (11). *Thus equation* (11) *is completely integrable.* Hence Theorem 2 is proved.

PART II.

4. **Necessary and sufficient conditions for invariance.** In Part I of this paper we considered a sufficient condition for the invariance of $f[y(\overset{1}{\underset{0}{\tau}}), y'(\overset{1}{\underset{0}{\tau}})]$ under a given arbitrary Volterra group of transformations (1). The task of finding necessary and sufficient conditions for the invariance of any functional $f[y(\overset{1}{\underset{0}{\tau}}), y'(\overset{1}{\underset{0}{\tau}})]$ with a Volterra variation and with continuous partial functional derivatives appears to be a difficult one. However, we shall see in this section that equation (11) is a necessary and sufficient condition in certain special cases.

Our discussion will necessitate the use of a new concept which we proceed to define.

Definition of a skew functional. If $S[y(\overset{1}{\underset{0}{\tau}}), y_1(\overset{1}{\underset{0}{\tau}}), y_2(\overset{1}{\underset{0}{\tau}}), \cdots, y_k(\overset{1}{\underset{0}{\tau}})/s]$ is a functional of the $k+1$ continuous functional arguments $y(\tau), y_1(\tau), \cdots, y_k(\tau)$ which depends on the point s and satisfies the functional equation

$$\int_0^1 y(s)\, S[y(\overset{1}{\underset{0}{\tau}}), y_1(\overset{1}{\underset{0}{\tau}}), y_2(\overset{1}{\underset{0}{\tau}}), \cdots, y_k(\overset{1}{\underset{0}{\tau}})/s]\, ds = 0,$$

then $S[y(\overset{1}{\underset{0}{\tau}}), y_1(\overset{1}{\underset{0}{\tau}}), \cdots, y_k(\overset{1}{\underset{0}{\tau}})/s]$ is defined to be a *skew functional* of $y(\tau), y_1(\tau), \cdots, y_k(\tau)$.

We shall have occasion to make use of the results of the following two lemmas.

LEMMA 1. *A necessary and sufficient condition that a functional* $G[y(\overset{1}{\underset{0}{\tau}})/s]$ *with a Volterra expansion**

$$(18)\quad g_0(s)+\sum_{i=1}^{\infty}\frac{1}{(i+1)!}\int_0^1\int_0^1\cdots\int_0^1 g_i(s,t_1,t_2,\cdots,t_i)y(t_1)y(t_2)\cdots y(t_i)\,dt_1\,dt_2\cdots dt_i$$

(all g_i*'s assumed continuous and* $|g_i|<\varkappa$, $|y|<\mu$, $\varkappa>0$, $\mu>0$*) be a skew functional of* $y(\tau)$† *is that all the* g_i*'s be skew symmetric functions of their arguments.*

LEMMA 2. *A necessary and sufficient condition that a linear functional*

$$(19)\quad l_0(s)+\int_0^1 l_1(s,t)y(t)\,dt+\int_0^1 l_2(s,t)y'(t)\,dt$$

(l_0, l_1, l_2 *are assumed continuous functions of their arguments and* $\dfrac{\partial l_2(s,t)}{\partial t}$ *is assumed to exist and to be continuous in* s *and* t*) be a skew functional of* $y(\tau)$ *and* $y'(\tau)$ *is that it reduce, by an integration by parts, to a skew linear functional of* $y(\tau)$.

Consider the proof of Lemma 1. Since (18) and its functional derivatives are evidently uniformly convergent series, integrations term by term and functional differentiations term by term of such series are valid. We have then

$$\int_0^1 y(s)\,G[y(\overset{1}{\underset{0}{\tau}})/s]\,ds=\int_0^1 y(s)g_0(s)\,ds+\sum_{i=1}^{\infty}\frac{1}{(i+1)!}\int_0^1\int_0^1\cdots\int_0^1 g_i(s,t_1,t_2,\cdots,t_i)$$
$$(20)\qquad \times y(t_1)y(t_2)\cdots y(t_i)y(s)\,dt_1\cdots dt_i\,ds\equiv 0 \text{ in } y.$$

Differentiating (20) functionally with respect to y at the point ξ_1, we get

$$g_0(\xi_1)+\sum_{i=1}^{\infty}\frac{1}{(i+1)!}\int_0^1\int_0^1\cdots\int_0^1\sum_{k=0}^{i} g_i(t_k,t_1,t_2,\cdots,t_{k-1},\xi_1,t_{k+1},t_{k+2},\cdots,t_i)$$
$$(20')\qquad \times y(t_1)y(t_2)\cdots y(t_i)\,dt_1\,dt_2\cdots dt_i\equiv 0 \text{ in } y,$$

---

* We can, without any loss of generality, assume that all the functions $g_i(s,t_1,t_2,\cdots,t_i)$ are respectively symmetric in their last i arguments $t_1,t_2,\cdots,t_i$.

† We note that Lemma 1 holds true even if we restrict ourselves to functions $y(\tau)$ which have continuous derivatives up to a certain order and which, with all their derivatives, vanish at 0 and 1. Evidently Lemma 1 includes the fundamental lemma of the calculus of variations as a special case.

where $[g_i]_{k=0} \equiv g_i(\xi_1, t_1, t_2, \cdots, t_i)$. In particular putting $y \equiv 0$, we get $g_0(\xi_1) = 0$. Differentiating (20′) functionally with respect to y and then putting $y \equiv 0$, we get $g_1(\xi_1, \xi_2) + g_1(\xi_2, \xi_1) = 0$. If we continue such a series of successive functional differentiations followed by putting $y \equiv 0$, we shall get

$$g_{i-1}(\xi_1, \xi_2, \xi_3, \cdots, \xi_i) + g_{i-1}(\xi_2, \xi_1, \xi_3, \cdots, \xi_i) + \cdots$$
$$\cdots + g_{i-1}(\xi_i, \xi_1, \xi_2, \cdots, \xi_{i-1}) = 0 \qquad (i = 1, 2, 3, \cdots).$$

Since $\xi_1, \xi_2, \xi_3, \cdots, \xi_i$ are arbitrary in the functional differentiations it follows that the above relations are true for all $\xi_1, \xi_2, \cdots, \xi_i$ in the interval 0, 1. Since these conditions are evidently also sufficient for (20) to hold, Lemma 1 has been proved.

We now proceed with the proof of Lemma 2. By hypothesis we have

$$\int_0^1 l_0(s)y(s)\,ds + \int_0^1\int_0^1 l_1(s, t)y(s)y(t)\,ds\,dt + \int_0^1\int_0^1 l_2(s, t)y(s)y'(t)\,ds\,dt \equiv 0.$$

An integration by parts yields

$$\int_0^1 l_0(s)y(s)\,ds + \int_0^1\int_0^1 \left\{l_1(s, t) - \frac{\partial l_2(s, t)}{\partial t}\right\} y(s)y(t)\,ds\,dt$$
$$+ \left[y(t)\int_0^1 l_2(s, t)y(s)\,ds\right]_{t=0}^{1} \equiv 0.$$

Consequently

$$(21) \qquad \int_0^1 l_0(s)y(s)\,ds + \int_0^1\int_0^1 \left\{l_1(s, t) - \frac{\partial l_2(s, t)}{\partial t}\right\} y(s)y(t)\,ds\,dt = 0$$

for all y with $y(0) = y(1) = 0$. Hence, by Lemma 1, $l_0(s) \equiv 0$ and and $\left\{l_1(s, t) - \frac{\partial l_2(s, t)}{\partial t}\right\}$ is a skew-symmetric function of s and t. Thus (21) is true for all continuous functions y. It follows therefore that

$$\left[y(t)\int_0^1 l_2(s, t)y(s)\,ds\right]_0^1 = 0$$

for all y. A necessary and sufficient condition that this be true is that

$$l_2(s, 0) \equiv l_2(s, 1) \equiv 0.$$

Thus Lemma 2 follows readily.

If we let $A[y(\tau), y'(\tau)/s]$ stand for an arbitrary continuous skew functional of $y(\tau)$ and $y'(\tau)$, then a necessary and sufficient condition that (6) hold is that

$$(22) \qquad f_{y'}[y, y'/s] = -\int_s^1 f_y[y, y'/r]\,\frac{H(r,s)}{H(s,s)}\,dr - \int_s^1 f_{y'}[y, y'/r]\,\frac{H_1(r,s)}{H(s,s)}\,dr + \frac{A[y, y'/s]}{H(s,s)}.$$

(22) is a Volterra integral equation of the second kind with the continuous kernel $-H_1(r,s)/H(s,s)$. The unique continuous solution $f_{y'}[y, y'/s]$ of this Volterra equation is given by

$$(22') \qquad (22')\ f_{y'}[y, y'/s] = \theta[y, y'/s] - \int_s^1 k(s,r)\,\theta[y, y'/r]\,dr,$$

where

$$\theta[y, y'/s] = -\int_s^1 f_y[y, y'/r]\,\frac{H(r,s)}{H(s,s)}\,dr + \frac{A[y, y'/s]}{H(s,s)}$$

and where $k(s,r)$ is the reciprocal kernel of $-H_1(r,s)/H(s,s)$. We can transform (22′) into the form

$$(22'') \qquad f_{y'}[y, y'/s] = \int_0^1 L(s,r) f_y[y, y'/r]\,dr + A[y, y'/s]/H(s,s) - \int_s^1 \frac{k(s,r)}{H(r,r)}\,A[y, y'/r]\,dr,$$

where $L(s,r)$ is the same function found in equation (11). In the cases in which it is impossible to find a skew functional $A[y, y'/s] \not\equiv 0$ that satisfies the integrability conditions* of equation (22″), it is seen readily that equation (11) will constitute a necessary and sufficient condition for the invariance of $f[y, y']$.

As a simple illustration of this result, consider the problem of finding a functional $f[y, y']$ of the form

* Following P. Lévy's analysis of integrability conditions for functional equations, we can exhibit at this point the integrability conditions for equation (22″); but we omit them for brevity's sake.

$$(23)\qquad f_{00}+\int_0^1 f_{10}(t)\,y(t)\,dt+\int_0^1 f_{01}(t)\,y'(t)\,dt$$
$$+\frac{1}{2!}\Big[\int_0^1\int_0^1 f_{20}(t_1,t_2)\,y(t_1)\,y(t_2)\,dt_1\,dt_2+2\int_0^1\int_0^1 f_{11}(t_1,t_2)\,y(t_1)\,y'(t_2)\,dt_1\,dt_2$$
$$+\int_0^1\int_0^1 f_{02}(t_1,t_2)\,y'(t_1)\,y'(t_2)\,dt_1\,dt_2\Big]$$

that admits the Volterra group of transformations

$$(23')\qquad \delta y(x)=\Big[\int_0^x y(s)\,ds\Big]\delta a.$$

We assume that all the f_{ik}'s are continuous; all the second order partial derivatives of f_{20}, f_{11}, f_{02} exist and are continuous; and, without any loss of generality, that f_{20} and f_{02} are symmetric functions. Since the first partial functional derivatives of (23) are linear functionals of $y(\tau)$ and $y'(\tau)$, we can apply Lemma 2 to equation (6). Doing so, functional equation (22″) becomes

$$(23'')\qquad f_{y'}[y,y'/s]=\int_0^1 L(s,r)\,f_y[y,y'/r]\,dr+A[\overset{1}{\underset{0}{y/s}}]$$

in which

$$L(s,r)=-1,\qquad r\geqq s$$
$$=0,\qquad r<s$$
$$A[\overset{1}{\underset{0}{y/s}}]=\int_0^1 a_1(s,t)\,y(t)\,dt$$
$$a_1(s,t)+a_1(t,s)=0$$

and $\dfrac{\partial a_1(s,t)}{\partial s}$, $\dfrac{\partial a_1(s,t)}{\partial t}$ exist and are continuous.

Let

$$H[\delta y(t)/s]=-\int_s^1 \delta y(t)\,dt$$
$$L[\delta y(r)/t]=\int_0^1 a_1(t,r)\,\delta y(r)\,dr.$$

Let $\mathfrak{H}$ be the adjoint linear functional of H and $\mathfrak{L}$ the adjoint linear functional of L. Evidently

$$\mathfrak{H}[\delta y(t)/s]=-\int_0^s \delta y(t)\,dt$$
$$\mathfrak{L}[\delta y(r)/t]=\int_0^1 a_1(r,t)\,\delta y(r)\,dr.$$

Making use of Levy's analysis* and following out a line of reasoning analogous to the one employed in getting the integrability conditions for equation (11), we find that the integrability condition for equation (23″) is

$$H[\mathfrak{L}[\delta y(r)/t]/s] \equiv L[\mathfrak{H}[\delta y(r)/t]/s]$$

which can be reduced to the form

$$\int_s^1 a_1(r, t)\,dt = \int_r^1 a_1(s, t)\,dt.$$

Hence in order that the integrability conditions may be satisfied, it is necessary that $a_1(r, t)$ satisfy the condition

$$\frac{\partial a_1(r, s)}{\partial r} - \frac{\partial a_1(s, r)}{\partial s} = 0$$

and, since $a_1(r, s)$ is a skew symmetric function, it is necessary that

$$\frac{\partial a_1(r, s)}{\partial r} + \frac{\partial a_1(r, s)}{\partial s} = 0.$$

But this differential equation can not be satisfied by a skew symmetric function $a_1(r, s) \not\equiv 0$ and vanishing for $r = 1$. Hence we have the result that a necessary and sufficient condition that a functional (23) admit the Volterra group of infinitesimal transformations (23′) is that it satisfy the completely integrable equation (11).

If the functional $f[y, y']$ is a linear functional of the form

$$f_{00} + \int_0^1 f_{10}(t)\,y(t)\,dt + \int_0^1 f_{01}(t)\,y'(t)\,dt \tag{24}$$

($f_{10}(t)$ and $f_{01}(t)$ assumed continuous in interval 0, 1.) the two functional derivatives $f_y[y, y'/s]$ and $f_{y'}[y, y'/s]$ will be continuous point functions. Consequently the expression in brackets in (6) will be a continuous point function. This result enables us to state the following evident but important theorem.†

* P. Lévy, Leçons d'analyse fonctionnelle, Part II, Chapt. 4.

† The importance of this theorem is brought out in a paper by the author "Invariant functionals, with continuity order p, of Fredholm and Volterra transformation groups", which has been accepted by the Bulletin of the American Mathematical Society.

THEOREM 3. *A necessary and sufficient condition that a linear functional* (24) *admit a given arbitrary Volterra group of transformations* (1) *is that it satisfy the completely integrable functional equation* (11).

5. Volterra expansions and calculation of solutions of equation (11). In order to obtain explicitly the invariant functionals that are solutions of the functional equation (11), we assume throughout this section of our paper that $f[y(\overset{1}{\underset{0}{\tau}}), y'(\overset{1}{\underset{0}{\tau}})]$ is developable in a Volterra expansion*

$$(25)\quad f_{00}+\sum_{j=1}^{\infty}\left\{\frac{1}{j!}\int_0^1\int_{\substack{0\\(j)}}^1\cdots\int_0^1\left[\sum_{k=0}^{j}\binom{j}{k}f_{j-k,k}(t_1, t_2, \cdots, t_{j-k}; t_{j-k+1}, \cdots, t_j)\right.\right.$$
$$\left.\left.\times\prod_{i=1}^{j-k}y(t_i)\prod_{i=j-k+1}^{j}y'(t_i)\right]dt_1\,dt_2\cdots dt_j\right\}.$$

We assume that $f_{j-k,k}$ is continuous in all its j arguments and, without any loss of generality, symmetric separately in the two sets of arguments $t_1, t_2, \cdots, t_{j-k}$ and $t_{j-k+1}, \cdots, t_j$. For convenience we assume also that

$$(25')\qquad |f_{j-k,k}|<\gamma,\qquad |y|<\varrho_1,\qquad |y'|<\varrho_2,$$

where γ, ϱ_1, ϱ_2 are positive constants. Obviously, then, (25) will be an absolutely and a uniformly convergent series.

Consider the problem of finding explicitly the solutions of equation (11); i. e., of finding the Volterra expansion of the integro-differential expressions $f[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}]$ invariant under the group of transformations (1). Assume a solution

$$(26)\qquad \Lambda[y, z]=\lambda_{00}+\sum_{j=1}^{\infty}\left\{\frac{1}{j!}\int_0^1\int_{\substack{0\\(j)}}^1\cdots\int_0^1\left[\sum_{k=0}^{j}\binom{j}{k}\right.\right.$$
$$\left.\left.\times\lambda_{j-k,k}(t_1, \cdots, t_{j-k}; t_{j-k+1}, \cdots, t_j)\prod_{i=1}^{j-k}y(t_i)\prod_{i=j-k+1}^{j}z(t_i)\right]dt_1\,dt_2\cdots dt_j\right\}$$

of the functional equation

$$(27)\qquad \Lambda_z[y, z/s]=\int_0^1 L(s, r)\,\Lambda_y[y, z/r]\,dr.$$

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* This is a generalization of Taylor's series given by Volterra. See for instance his "Leçons sur les équations integrales", 1913.

Thus we shall have

$$\mathcal{A}_y[y, z/r]$$

$$= \sum_{j=1}^{\infty} \left\{ \frac{1}{j!} \int_0^1 \int_{0\,(j-1)}^{1} \cdots \int_0^1 \left[\sum_{k=0}^{j-1} \binom{j}{k} (j-k)\, \lambda_{j-k,k}(r,\ t_2, \cdots, t_{j-k};\ t_{j-k+1}, \cdots, t_j) \right.\right.$$

$$\left.\left. \times \prod_{i=2}^{j-k} y(t_i) \prod_{i=j-k+1}^{j} z(t_i) \right] dt_2\, dt_3 \cdots dt_j \right\}.$$

Since

$$\frac{1}{j!}\ \frac{j!\,(j-k)}{k!\,(j-k)!} = \frac{1}{(j-1)!}\ \frac{(j-1)!}{k!\,(j-k-1)!},$$

we have

$$\mathcal{A}_y[y, z/r]$$

$$= \sum_{j=1}^{\infty} \left\{ \frac{1}{(j-1)!} \int_0^1 \int_{0\,(j-1)}^{1} \cdots \int_0^1 \left[\sum_{k=0}^{j-1} \binom{j-1}{k} \lambda_{j-k,k}(r, t_1, \cdots, t_{j-k-1};\ t_{j-k}, \cdots, t_{j-1}) \right.\right.$$

$$\left.\left. \times \prod_{i=1}^{j-k-1} y(t_i) \prod_{i=j-k}^{j-1} z(t_i) \right] dt_1\, dt_2 \cdots dt_{j-1} \right\}.$$

We have also

(28) $\mathcal{A}_z[y, z/s]$

$$= \sum_{j=1}^{\infty} \left\{ \frac{1}{j!} \int_0^1 \int_{0\,(j-1)}^{1} \cdots \int_0^1 \left[\sum_{k=1}^{j} \binom{j}{k} k\, \lambda_{j-k,k}(t_1, \cdots, t_{j-k};\ t_{j-k+1}, \cdots, t_{j-1}, s) \right.\right.$$

$$\left.\left. \times \prod_{i=1}^{j-k} y(t_i) \prod_{i=j-k+1}^{j-1} z(t_i) \right] dt_1\, dt_2 \cdots dt_{j-1} \right\}$$

$$= \sum_{j=1}^{\infty} \left\{ \frac{1}{(j-1)!} \int_0^1 \int_{0\,(j-1)}^{1} \cdots \int_0^1 \left[\sum_{k=1}^{j} \binom{j-1}{k-1} \lambda_{j-k,k}(t_1, \cdots, t_{j-k};\ t_{j-k+1}, \cdots, t_{j-1}, s) \right.\right.$$

$$\left.\left. \times \prod_{i=1}^{j-k} y(t_i) \prod_{i=j-k+1}^{j-1} z(t_i) \right] dt_1\, dt_2 \cdots dt_{j-1} \right\}$$

$$= \sum_{j=1}^{\infty} \left\{ \frac{1}{(j-1)!} \int_0^1 \int_{0\,(j-1)}^{1} \cdots \int_0^1 \left[\sum_{l=0}^{j-1} \binom{j-1}{l} \lambda_{j-1-l,\,l+1}(t_1, \cdots, t_{j-1-l};\ t_{j-l}, \cdots, t_{j-1}, s) \right.\right.$$

$$\left.\left. \times \prod_{i=1}^{j-1-l} y(t_i) \prod_{i=j-l}^{j-1} z(t_i) \right] dt_1\, dt_2 \cdots dt_{j-1} \right\}, \quad \text{where } k = l+1.$$

Thus

$$(29)\quad \int_0^1 L(s,r)\,\mathcal{A}_y[y, z/r]\,dr$$

$$= \sum_{j=1}^{\infty}\left\{\frac{1}{(j-1)!}\int_0^1 \underset{(j)}{\int_0^1\cdots\int_0^1}\left[\sum_{k=0}^{j-1}\binom{j-1}{k} L(s,r)\,\lambda_{j-k,k}(r, t_1, \cdots, t_{j-k-1}; t_{j-k}, \cdots, t_{j-1})\right.\right.$$

$$\left.\left.\times \prod_{i=1}^{j-k-1} y(t_i) \prod_{i=j-k}^{j-1} z(t_i)\right] dt_1\, dt_2 \cdots dt_{j-1}\, dr\right\}.$$

Substituting (28) and (29) in (27) and equating coefficients of similar terms in y and z, we get the recurrence formula

$$\lambda_{j-1-l,l+1}(t_1, \cdots, t_{j-1-l}; t_{j-l}, \cdots, t_{j-1}, s)$$

$$= \int_0^1 L(s,r)\,\lambda_{j-l,l}(r, t_1, \cdots, t_{j-1-l}; t_{j-l}, \cdots, t_{j-1})\,dr,$$

$$(L(s,r) = 0 \text{ when } r < s)$$

or

$$(30)\quad \lambda_{i,k}(t_1, \cdots, t_i; t_{i+1}, \cdots, t_{i+k-1}, t_{i+k})$$

$$= \int_0^1 L(t_{i+k}, r)\,\lambda_{i+1,k-1}(r, t_1, \cdots, t_i; t_{i+1}, \cdots, t_{i+k-1})\,dr$$

$$(L(t_{i+k}, r) = 0 \text{ when } r < t_{i+k}).$$

Applying (30) until $\lambda_{i,k}$ is written in terms of λ's with second index zero, we get

$$\lambda_{i,k}(t_1, \cdots, t_i; t_{i+1}, \cdots, t_{i+k-1}, t_{i+k}) = \int_0^1 \underset{(k)}{\int_0^1\cdots\int_0^1} L(t_{i+k}, s_1)\,L(t_{i+k-1}, s_2)\cdots$$

$$\cdots L(t_{i+1}, s_k)\,\lambda_{i+k,0}(s_k, \cdots, s_1, t_1, \cdots, t_i)\,ds_1\,ds_2 \cdots ds_k$$

$$(L(t_{i+k-l}, s_{l+1}) = 0 \text{ when } s_{l+1} < t_{i+k-l})$$

which can be written as

$$(31)\quad \lambda_{i,k}(t_1, \cdots, t_i; t_{i+1}, \cdots, t_{i+k-1}, t_{i+k}) = \int_0^1 \underset{(k)}{\int_0^1\cdots\int_0^1} L(t_{i+k}, s_k)\,L(t_{i+k-1}, s_{k-1})\cdots$$

$$\cdots L(t_{i+1}, s_1)\,\lambda_{i+k,0}(s_1, \cdots, s_k, t_1, \cdots, t_i)\,ds_1\,ds_2 \cdots ds_k$$

$$(L(t_{i+l}, s_l) = 0 \text{ when } s_l < t_{i+l}).$$

Substituting (31) in (26), we get

$$(32)\quad \mathcal{A}[y, z] = \lambda_{00} + \sum_{j=1}^{\infty}\left\{\frac{1}{j!}\int_0^1 \int_{\substack{0\\(j+k)}}^1 \cdots \int_0^1 \left[\sum_{k=0}^{j}\binom{j}{k} L(t_j, s_k)\, L(t_{j-1}, s_{k-1}) \cdots \right.\right.$$

$$\left.\left. \cdots L(t_{j-k+1}, s_1)\, \lambda_{j,0}(s_1, \cdots, s_k, t_1, \cdots, t_{j-k}) \prod_{i=1}^{j-k} y(t_i) \prod_{i=j-k+1}^{j} z(t_i)\right] dt_1\, dt_2 \cdots dt_j\, ds_1\, ds_2 \cdots ds_k\right\}$$

$$(L(t_{j-k+l}, s_l) = 0 \text{ when } s_l < t_{j-k+l})$$

Assume that

$$|L(s, r)| \leqq N, \qquad |\lambda_{j,0}| \leqq M_j,$$

$$\left.\begin{aligned} |y(\tau)| &< \varrho_1 \\ |z(\tau)| &< \varrho_2 \end{aligned}\right\} < \varrho_3 \quad \text{in (32).}$$

Assume further that $\mathcal{A}[y, 0]$ is absolutely convergent for $|y(\tau)| \leqq \varrho$; i. e., $\sum_{j=1}^{\infty} \frac{1}{j!} M_j \varrho^j$ is a convergent series. A dominant series for (32) (omitting λ_{00}) is given by

$$(33)\quad \sum_{j=1}^{\infty}\frac{1}{j!}\sum_{k=0}^{j}\binom{j}{k} N^k M_j \varrho_1^{j-k} \varrho_2^k < \sum_{j=1}^{\infty}\frac{1}{j!}\sum_{k=0}^{j}\binom{j}{k} N^k M_j \varrho_3^j$$

$$= \sum_{j=1}^{\infty}\frac{M_j \varrho_3^j}{j!}\sum_{k=0}^{j}\binom{j}{k} N^k = \sum_{j=1}^{\infty}\frac{1}{j!} M_j \varrho_3^j (N+1)^j.$$

Equation (33) will be convergent if $\varrho_3 \leqq \varrho/(N+1)$. Hence (32) will be absolutely convergent for $\varrho_3 \leqq \varrho/(N+1)$.

$\mathcal{A}[y, z]$, as given by (32), will be the solution of the functional equation (27) if $\varrho_3 \leqq \varrho/(N+1)$, and will be such that, for $z(\tau) \equiv 0$,

$$\mathcal{A}[y, 0] \equiv \mathfrak{L}[y],$$

where $\mathfrak{L}[y]$ is arbitrary and is convergent for $|y| \leqq \varrho$.

We can apply this discussion to the problem on hand by taking $z(\tau) \equiv y'(\tau)$ and considering the functional equation (11). Hence it is immediate that the required $f[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}]$ can be written as

$$(34)\qquad f[y(\overset{1}{\underset{0}{\tau}}),\ y'(\overset{1}{\underset{0}{\tau}})] = f_{00} + \sum_{j=1}^{\infty}\left\{\frac{1}{j!}\int_0^1 \int_{\substack{0\\(j)}}^1 \cdots \int_0^1 \left[\sum_{k=0}^{j}\binom{j}{k}\right.\right.$$

$$\left.\left. \times f_{j-k,k}(t_1, \cdots, t_{j-k};\ t_{j-k+1}, \cdots, t_j) \prod_{i=1}^{j-k} y(t_i) \prod_{i=j-k+1}^{j} y'(t_i)\right] dt_1\, dt_2 \cdots dt_j\right\}$$

where

$$f_{j-k,k}(t_1, \cdots, t_{j-k}; t_{j-k+1}, \cdots, t_j)$$

$$= \int_0^1 \int_0^1 \cdots \int_0^1 L(t_j, s_k)\, L(t_{j-1}, s_{k-1}) \cdots L(t_{j-k+2}, s_2)\, L(t_{j-k+1}, s_1)$$

$$\times f_{j,0}(s_1, \cdots, s_k, t_1, \cdots, t_{j-k})\, ds_1\, ds_2 \cdots ds_k,$$

$$(L(t_{j-k+l}, s_l) = 0 \text{ when } s_l < t_{j-k+l}).$$

6. **Example of an invariant.** Take the group determined by the infinitesimal transformations

$$\delta y(x) = \left[\int_0^x y(s)\, ds\right] \delta a,$$

that is $H(x, s) = 1$. The kernel for the group of the finite transformations will be given by

$$K(x, s/a) = a + \frac{a^2 \int_s^x ds_1}{2!} + \frac{a^3 \int_s^x ds_2 \int_s^{s_2} ds_1}{3!} + \cdots$$

$$= \sum_{l=1}^{\infty} \frac{a^l (x-s)^{l-1}}{l!\,(l-1)!} = \frac{\sqrt{a(s-x)}}{s-x} J_1\left(2\sqrt{a(s-x)}\right),$$

where $J_1(z)$ is the Bessel coefficient of order 1.

In this case the functional equation (11) becomes

$$f_{y'}[\overset{1}{y}, \overset{1}{y}'/s] = \int_0^1 L(s, r) f_y[\overset{1}{y}, \overset{1}{y}'/r]\, dr,$$

where

$$L(s, r) = -1, \qquad r \geqq s,$$
$$= 0, \qquad r < s.$$

Equation (31) becomes

$$f_{i,k}(t_1, \cdots, t_i; t_{i+1}, \cdots, t_{i+k})$$

$$= (-1)^k \int_{t_{i+1}}^1 ds_1 \int_{t_{i+2}}^1 ds_2 \cdots \int_{t_{i+k}}^1 f_{i+k,0}(s_1, \cdots, s_k, t_1, \cdots, t_i)\, ds_k.$$

For initial condition $y'(\tau) \equiv 0$, take

$$(35)\quad f[y(\overset{1}{\tau}), 0] \equiv F[y(\overset{1}{\tau})]$$

$$= f_{00} + \int_0^1 f_{10}(s) y(s)\, ds + \frac{1}{2!} \int_0^1 ds \int_0^1 f_{20}(s, s') y(s) y(s')\, ds'.$$

Then

$$(36)\quad f[y(\overset{1}{\underset{0}{\tau}}),\ y'(\overset{1}{\underset{0}{\tau}})] = f_{00}+\int_0^1 f_{10}(t_1)\,y(t_1)\,dt_1$$
$$-\int_0^1 dt_1\int_{t_1}^1 f_{10}(s)\,y'(t_1)\,ds+\frac{1}{2!}\Big[\int_0^1 dt_1\int_0^1 f_{20}(t_1,\ t_2)\,y(t_1)\,y(t_2)\,dt_2$$
$$-\binom{2}{1}\int_0^1 dt_1\int_0^1 dt_2\int_{t_2}^1 f_{20}(s_1,\ t_1)\,y(t_1)\,y'(t_2)\,ds_1$$
$$+\binom{2}{2}\int_0^1 dt_1\int_0^1 dt_2\int_{t_1}^1 ds_1\int_{t_2}^1 f_{20}(s_1,\ s_2)\,y'(t_1)\,y'(t_2)\,ds_2\Big]$$

is the integro-differential functional of the form $f[y(\overset{1}{\underset{0}{\tau}}),\ y'(\overset{1}{\underset{0}{\tau}})]$ invariant under the Volterra group of transformations

$$y_1(x) = y(x)+\int_0^x \frac{\sqrt{a(s-x)}}{s-x}\,J_1\big(2\sqrt{a(s-x)}\big)\,y(s)\,ds$$

with the given initial condition (35).

A passing remark on integro-differential equations of the form

$$(37)\qquad f[y(\overset{1}{\underset{0}{\tau}}),\ y'(\overset{1}{\underset{0}{\tau}})] = 0$$

may not be out of place here. If $y(x)$ is a solution of (37), and if (37) is invariant under (1), then

$$y_1(x) = y(x)+\int_0^x K(x,\ s/a)\,y(s)\,ds$$

will also be a solution.

To illustrate this consider a particular example.

$$(38)\quad \int_0^1 y(t_1)\,dt_1-\int_0^1 (1-t_1)\,y'(t_1)\,dt_1+\frac{1}{2}\int_0^1 dt_1\int_0^1 y(t_1)\,y(t_2)\,dt_2$$
$$-\int_0^1 dt_1\int_0^1 (1-t_2)\,y(t_1)\,y'(t_2)\,dt_2$$
$$+\frac{1}{2}\int_0^1 dt_1\int_0^1 (1-t_1)\,(1-t_2)\,y'(t_1)\,y'(t_2)\,dt_2 = 0$$

is invariant under the group of transformations

$$y_1(x) = y(x)+\int_0^x \frac{\sqrt{a(s-x)}}{s-x}\,J_1\big(2\sqrt{a(s-x)}\big)\,y(s)\,ds.$$

Obviously $y = -2$ is a solution of (38). Hence

$$y(x) = -2 - 2\int_0^x \frac{\sqrt{a(s-x)}}{s-x} J_1(2\sqrt{a(s-x)})\, ds$$

is also a solution of (38).

If the upper limit of the integrals in (38) is a parameter u, then

$$y(x) = -\frac{2}{u} - \frac{2}{u}\int_0^x \frac{\sqrt{a(s-x)}}{s-x} J_1(2\sqrt{a(s-x)})\, ds$$

is a solution of (38) with u as upper limits of the integrals.

7. **Application to functionals $f[y(\overset{1}{\underset{0}{\tau}})]$.** A special application of the preceding theory* enables us to consider functionals $f[y(\overset{1}{\underset{0}{\tau}})]$ and for convenience we shall restrict ourselves to such functionals with a Volterra expansion.

In this case, equation (6) becomes

$$\int_0^1 y(s)\, ds \int_s^1 H(t,s) f_y[y(\overset{1}{\underset{0}{\tau}})/t]\, dt \equiv 0 \text{ in } y.$$

Hence

$$\int_s^1 H(t,s) f_y[y(\overset{1}{\underset{0}{\tau}})/t]\, dt = G[y(\overset{1}{\underset{0}{\tau}})/s],$$

where $G[y(\overset{1}{\underset{0}{\tau}})/s]$ is an arbitrary Volterra developable skew functional of y only. Evidently it is necessary that $G[y(\overset{1}{\underset{0}{\tau}})/1] \equiv 0$ and that $\partial G[y(\overset{1}{\underset{0}{\tau}})/s]/\partial s$ exist. By differentiation we have the Volterra integral equation

$$(39) \qquad f_y[y(\overset{1}{\underset{0}{\tau}})/s] = -\frac{\frac{\partial G[y/s]}{\partial s}}{H(s,s)} + \int_s^1 \frac{\frac{\partial H(t,s)}{\partial s}}{H(s,s)} f_y[y(\overset{1}{\underset{0}{\tau}})/t]\, dt$$

whose unique continuous solution (since $H(s,s) \neq 0$) is given by

$$(40) \qquad f_y[y(\overset{1}{\underset{0}{\tau}})/s] = -\frac{\frac{\partial G[y/s]}{\partial s}}{H(s,s)} + \int_s^1 R(s,t) \frac{\frac{\partial G[y/t]}{\partial t}}{H(t,t)}\, dt,$$

where $R(s,t)$ is the reciprocal kernel of $\frac{\partial H(t,s)}{\partial s} \Big/ H(s,s)$.

* We assume in this section that $\frac{\partial H(t,s)}{\partial s}$ exists and is continuous and that $H(s,s) \neq 0$; but we do not assume the existence of $\frac{\partial H(t,s)}{\partial t}$.

The integrability condition* for equation (40) is

$$\frac{\frac{\partial G_y[y/r, s]}{\partial r}}{H(r, r)} - \int_r^1 \frac{R(r, t)}{H(t, t)} \frac{\partial G_y[y/t, s]}{\partial t}\, dt$$
$$\equiv \frac{\frac{\partial G_y[y/s, r]}{\partial s}}{H(s, s)} - \int_s^1 \frac{R(s, t)}{H(t, t)} \frac{\partial G_y[y/t, r]}{\partial t}\, dt.$$

In the cases in which it is impossible to find a skew functional $G[y/s] \not\equiv 0$ that will satisfy this integrability condition, equation (39) will reduce to

$$f_y[y\,(\overset{1}{\underset{0}{\tau}})/s] = \int_s^1 \frac{\partial H(t, s)}{\partial s} \frac{1}{H(s, s)} f_y[y\,(\overset{1}{\underset{0}{\tau}})/t]\, dt.$$

Since $H(s, s) \neq 0$, it follows that $f_y[y\,(\overset{1}{\underset{0}{\tau}})/s] \equiv 0$. Hence, since

$$\delta f[y\,(\overset{1}{\underset{0}{\tau}})] = \int_0^1 f_y[y\,(\overset{1}{\underset{0}{\tau}})/t]\, \delta y(t)\, dt.$$

The functional $f[y\,(\overset{1}{\underset{0}{\tau}})]$ must be independent of $y(\tau)$.

That such cases actually exist we show by the following two simple illustrations. Consider functionals $f[y\,(\overset{1}{\underset{0}{\tau}})]$ of the form

$$f_0 + \int_0^1 f_1(t)\, y(t)\, dt + \int_0^1 \int_0^1 f_2(t_1, t_2)\, y(t_1)\, y(t_2)\, dt_1\, dt_2 \tag{41}$$

($f_1(t), f_2(t_1, t_2)$ assumed continuous) admitting the Volterra group of transformations

$$\delta y(x) = \left[H(x) \int_0^x y(s)\, ds\right] \delta a. \tag{42}$$

The skew functional $G[y/s]$ in this case is of the form $\int_0^1 g(s, t)\, y(t)\, dt$ and the integrability conditions reduce to

$$H(s) \frac{\partial g(r, s)}{\partial r} + H(r) \frac{\partial g(r, s)}{\partial s} = 0.$$

* Cf. Volterra's Leçons sur les fonctions de lignes, Chapt. II, p. 47. This result can also be deduced from P. Lévy's analysis of the integrability conditions of functional equations with ordinary functional derivatives. Cf. his Leçons d'analyse fonctionnelle, Part II, Chapt. I.

But it is impossible to find a skew symmetric function (not identically zero) which satisfies this differential equation and vanishes for $r = 1$. Hence from our previous theory we can conclude that there exist no functionals of the form (41), other than constants, admitting a Volterra group of transformations (42).

We state here the obvious but interesting theorem.

THEOREM 4. *There exist no linear functionals*

$$f_0 + \int_0^1 f_1(t)\, y(t)\, dt \qquad (f_1(t) \text{ continuous}),$$

other than constants, admitting a given arbitrary Volterra Group of transformations (1).*

It is to be noted that a parameter u could just as well have been taken for the upper limits (instead of 1) in the integrals throughout this paper.

* Cf. the note to the first sentence of this section.

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NEW PROOFS OF TWO WELL KNOWN THEOREMS ON QUADRATIC FORMS.

By J. F. Ritt.

Let N represent a prime positive integer. It is well known that if $N \equiv 1$, mod 4, it is of the form $\alpha^2 + \beta^2$, where α and β are integers, and that if $N \equiv 1$, mod 6, it is of the form $\alpha^2 + \alpha\beta + \beta^2$. The present paper proves these facts by an analytic method, which involves the construction of a Riemann surface and its uniformization with elliptic functions. We do not think that such a method has been used before in the theory of numbers.

This method can also be used for determining the number of distinct representations, but as the mere existence proofs sufficiently illustrate the method, and as no new results are in the balance, we shall not go into these determinations.

First let N be a prime of the form $4n+1$. There exists an integer r which belongs to the exponent 4, modulo N. We are going to perform substitutions upon the indices $1, 2, \cdots, N$. Consider the three substitutions given by the congruences

$$(1) \qquad \nu' \equiv r\nu, \qquad \nu' \equiv r\nu + 1, \qquad \nu' \equiv r^2\nu - r^2, \qquad \text{mod } N.$$

We represent these substitutions by s_1, s_2, s_3 respectively. Both s_1 and s_2 are of order 4, while s_3 is of order 2. Each of the substitutions leaves a single index fixed.

As $s_1^2 s_2$ has the representation $\nu' \equiv \nu + 1$, mod N, the three substitutions generate a transitive group. Furthermore, $s_1 s_2 s_3 = 1$. It is therefore possible to construct a Riemann surface of N sheets, which has for its only critical points three arbitrarily assigned points a, b and c, at which the N sheets undergo the substitutions s_1, s_2, s_3 respectively.

We shall find the genus of this surface. At a and at b we have $(N-1)/4$ branch points of order 3, whereas at c we have $(N-1)/2$ simple branch points. The sum of the orders of all the branch points is $2N-2$, so that the genus is zero.

Consequently the surface is that of the inverse, $\Phi^{-1}(w)$, of a rational function $\Phi(z)$. We may assume that $\Phi^{-1}(w)$ assumes at those three places on its surface, at a, b, and c, where it is uniform, the values a, b and c respectively. This can always be brought about by replacing $\Phi^{-1}(w)$ by a linear function of itself.

We shall uniformize $\Phi^{-1}(w)$ by means of $\wp(z)$ with $g_3 = 0$. It is well known that in this lemniscatic case, $e_1 = -e_3$, and $e_2 = 0$. The function $\wp^2(z)$ assumes the values 0, ∞ and e_1^2 respectively 4, 4 and 2 times each, wherever it assumes them. These are the only multiple values of $\wp^2(z)$.

We now attribute to a, b and c the values 0, ∞ and e_1^2 respectively, and consider the N values of $\Phi^{-1}[\wp^2(z)]$. These N values are each uniform *im kleinen*, and hence also *im großen*. They are N meromorphic functions; let $f(z)$ represent some one of them. As $\Phi^{-1}(w)$ assumes the values 0, ∞ and e_1^2 only at places where it is uniform, $f(z)$ assumes these values respectively 4, 4 and 2 times each, wherever it assumes them.

Let $p(z)$ represent the inverse of $\wp^2(z)$. The critical points of $p(z)$ are 0 and ∞, at which its branches are all permuted in fours, and e_1^2, where its branches are permuted in pairs. Consider the values of $p[f(z)]$. They are all uniform *im kleinen*, and hence also *im großen*. Let $\beta(z)$ represent one of these meromorphic functions. We have

$$\wp^2(z) = \Phi\{\wp^2[\beta(z)]\}. \tag{2}$$

If $\beta(z)$ had a pole with a finite affix, $\wp^2(z)$ would have an essential singularity with a finite affix. Hence $\beta(z)$ is entire. We see from (2) that $\wp[\beta(z)]$, being algebraically related to $\wp^2(z)$, is an elliptic function. Its derivative,

$$\wp'[\beta(z)]\,\beta'(z)$$

must therefore be an elliptic function. But as $\wp'[\beta(z)]$ is an elliptic function, because of its algebraic relation to $\wp[\beta(z)]$, $\beta'(z)$ is also elliptic. As $\beta'(z)$ is entire, it must be a constant. Hence $\beta(z)$ is linear.

Let $uz+v$ be the inverse of $\beta(z)$. Then (2) becomes

$$\wp^2(uz+v) = \Phi[\wp^2(z)]. \tag{3}$$

We may suppose that 1 and i are a primitive pair of periods for $\wp^2(z)$. Then $1/u$ and i/u are a primitive pair for $\wp^2(uz+v)$. But as every period of $\wp^2(z)$ is also a period of the first member of (3), we have

$$1 = \frac{\alpha}{u} + \frac{\beta i}{u},$$

where α and β are integers. Hence $u = \alpha + \beta i$.

Thus a primitive period parallelogram for $\wp^2(z)$ has an area $\alpha^2+\beta^2$ times that of a primitive parallelogram for $\wp^2(uz+v)$. It is well known, and easy to see, that this ratio is the degree of $\Phi(z)$ in (3). Hence

$$N = \alpha^2+\beta^2,$$

which was to be proved.

To prove that a prime N of the form $6n+1$ has a representation $\alpha^2+\alpha\beta+\beta^2$, we take an integer r which belongs to the exponent 6, modulo N, and consider the substitutions

$$\nu' \equiv r\nu, \qquad \nu' \equiv r^2\nu+1, \qquad \nu' \equiv r^3\nu-r^3, \qquad \text{mod } N,$$

which are of orders 6, 3 and 2 respectively. We construct a Riemann surface as above. Then we take $\wp(z)$ with $g_2 = 0$. It is well known that, in this equianharmonic case, $e_1^3 = e_2^3 = e_3^2$. Also, $\wp^3(z)$ assumes the values ∞, 0 and e_1^3, respectively 6, 3 and 2 times each, wherever it assumes them. As periods we take 1 and $e^{2\pi i/3}$. The rest of the work is similar to that for $N = 4n+1$.

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SOME RELATIONS BETWEEN COMPOUND DETERMINANTS.

By Professor W. H. Metzler.

In 1905 Muir* gave a relation (2′ below) between two types of compound determinants of orders n and $n-1$, which we shall denote by $D_{[n,n-1]}$ and $\Delta_{[n-1,n]}$ respectively. The elements of $D_{[n,n-1]}$ are minors of order $(n-1)$ formed from n arrays, $A_1, A_2, \cdots, A_n$ of $(n-1)$ rows and n columns each; and the elements of $\Delta_{[n-1,n]}$ are determinants of order n formed by combining a row of A_n with each of the other A's, i. e., the element in the position (i, j) is the determinant formed by combining the jth row of A_n with A_i.

In 1913 the writer† gave a type (3a below) for $D_{[\lambda,l]}$ which resulted in having the corresponding $\Delta_{[\varkappa,n]}$ break up into factors.

In 1914 the writer‡ generalized Muir's result giving a relation (2 below) between $D_{[\lambda,l]}$ of order λ and $\Delta_{[\varkappa,n]}$ of order $\varkappa$, where $\varkappa < \lambda$ and $\lambda \equiv (n)_l$. The elements of $D_{[\lambda,l]}$ being now minors of order l formed from the element of the λ arrays $A_1, A_2, \cdots, A_\lambda$ of l rows and n columns each; and the elements of $\Delta_{[\varkappa,n]}$ being determinants of order n formed by combining the arrays $A_1, A_2, \cdots, A_\varkappa$ with the arrays $B_1, B_2, \cdots, B_\varkappa$ in all possible ways, where $B_1, B_2, \cdots, B_\varkappa$ are the $\varkappa$ arrays of $(n-l)$ rows and n columns formed from the rows, taken $(n-l)$ at a time, of the arrays $A_{\varkappa+1}, \cdots, A_\lambda$ and h of those of $A_\varkappa$, which are exclusive of those which are complementary to $A_{\varkappa+1}, \cdots, A_\lambda$, and where $(\lambda-\varkappa)l+h = n$ and $h<l$.

In 1918 Muir§ gave a relation (4a below) between $\Delta_{[n,n]}$ and $\Delta_{[n-1,n]}$. In Sept. 1918 Whittaker‖ gave the relation between $D_{[n,n-1]}$ and $\Delta_{[n-1,n]}$ which was given by Muir thirteen years earlier and made the fresh advance of showing that these two forms had as an equivalent a simple determinant of order n^2 which we may denote by $A_{[n,n-1]}$ of the form

$$\begin{vmatrix} (A_1) & (0) & \cdots & (0) \\ (0) & (A_2) & \cdots & (0) \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \\ (0) & (0) & \cdots & (A_n) \\ (1) & (1) & \cdots & (1) \end{vmatrix}$$

* Messenger of Math., new ser., No. 416, Dec. 1905. Referred to hereafter as Muir, Paper I.

† Metzler: Proc. Roy. Soc. Edinburgh, vol. 34, Part 1. Hereafter referred to as Metzler, Paper I.

‡ Ann. of Math., 2nd ser., vol. 15, No. 4, June 1914. Hereafter referred to as Metzler, Paper II.

§ Muir: Messenger of Math., new ser., No. 564, vol. 47, April 1918. Hereafter referred to as Muir, Paper II.

‖ Whittaker: Proc. Edin. Math. Soc., vol. 36, Part 2.

where (A_i) denotes an array of $(n-1)$ rows and n columns; (1) denotes a determinant of order n with ones for elements along the principal diagonal and all the other elements zero; and where (0) denotes an array of $(n-1)$ rows and n columns of zeros.

In 1920 Muir* gave a relation between forms similar to $A_{[n,n-1]}$ and $\Delta_{[n,n]}$ or which come out of the relation between these by giving some of the elements special values.

The principal object of this paper is to bring all of these relations (except those involving $A_{[n,n-1]}$) under one general theorem which includes them as special cases.

Let $x_1, x_2, \cdots, x_\lambda$ be the λ combinations of any n numbers, denoted by X, taken l at a time, and let $l \cdot \lambda$ integers, denoted by A, be broken up into λ sets of l in each and let these sets be represented by $\alpha_1, \alpha_2, \cdots, \alpha_\lambda$.

Let the compound determinant $D_{[\lambda,l]}$ of order λ with elements of order l be formed from a rectangular array of n rows and $l \cdot \lambda$ columns of elements, thus

$$D_{[\lambda,l]} \equiv \begin{vmatrix} \begin{vmatrix} x_1 \\ \alpha_1 \end{vmatrix} & \begin{vmatrix} x_2 \\ \alpha_1 \end{vmatrix} & \cdots & \begin{vmatrix} x_\lambda \\ \alpha_1 \end{vmatrix} \\ \begin{vmatrix} x_1 \\ \alpha_2 \end{vmatrix} & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \begin{vmatrix} x_1 \\ \alpha_\lambda \end{vmatrix} & & & \end{vmatrix}.$$

Where writing but the first row and first column sufficiently indicates the law of formation; and where $\begin{vmatrix} x_i \\ \alpha_j \end{vmatrix}$ denotes the minor of order l formed by the elements of the array common to the ith selection of rows and the jth selection of colums.

Let

$$D'_{[\lambda,n-l]} \equiv \begin{vmatrix} \begin{vmatrix} x'_1 \\ \beta_1 \end{vmatrix} & \begin{vmatrix} x'_2 \\ \beta_1 \end{vmatrix} & \cdots & \begin{vmatrix} x'_\lambda \\ \beta_1 \end{vmatrix} \\ \begin{vmatrix} x'_1 \\ \beta_2 \end{vmatrix} & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \begin{vmatrix} x'_1 \\ \beta_\lambda \end{vmatrix} & & & \end{vmatrix}.$$

* Muir: Quart. Journ. of Math., vol. 48, No. 192, 1920. Referred to hereafter as Muir, Paper III.

Where x_i' is the complementary of the combination x_i with respect to X; and where $\beta_1, \beta_2, \cdots, \beta_\lambda$ are the combinations of any set of n numbers, denoted by B, taken $(n-l)$ at a time.

Then the product

$$D_{[\lambda,l]}\, D'_{[\lambda,n-l]} = \begin{vmatrix} \left|\begin{matrix} X \\ \alpha_1 \beta_1 \end{matrix}\right| & \left|\begin{matrix} X \\ \alpha_1 \beta_2 \end{matrix}\right| & \cdots & \left|\begin{matrix} X \\ \alpha_1 \beta_\lambda \end{matrix}\right| \\ \left|\begin{matrix} X \\ \alpha_2 \beta_1 \end{matrix}\right| & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \left|\begin{matrix} X \\ \alpha_\lambda \beta_1 \end{matrix}\right| & & & \end{vmatrix} \equiv \Delta_{[\lambda,n]}.$$

But $D'_{[\lambda,n-l]}$ being the $(n-1)$th compound of $\left|\begin{matrix} X \\ \beta \end{matrix}\right|$ is equal to $\left|\begin{matrix} X \\ B \end{matrix}\right|^{(n-1)_l}$. Therefore*

$$D_{[\lambda,l]} \cdot \left|\begin{matrix} X \\ B \end{matrix}\right|^{(n-1)_l} = \Delta_{[\lambda,n]}. \tag{1}$$

Instead of the multiplier $D'_{[\lambda,n-l]}$ use $D''_{[\lambda,n-l]}$ obtained from $D'_{[\lambda,n-l]}$ by replacing the n numbers B by the n numbers C where C contains h of those of B and $(n-h)$ of those of A; and where the $(n-h)$ numbers of A are those in $\alpha_{x+1}, \alpha_{x+2}, \cdots, \alpha_\lambda$. Let $\gamma_{x+1}, \gamma_{x+2}, \cdots, \gamma_\lambda$ be the complementaries of $\alpha_{x+1}, \alpha_{x+2}, \cdots, \alpha_\lambda$ respectively with respect to C. Then the product

$$D_{[\lambda,l]} \cdot D''_{[\lambda,n-l]} = \begin{vmatrix} \left|\begin{matrix} X \\ \alpha_1 \gamma_1 \end{matrix}\right| & \left|\begin{matrix} X \\ \alpha_1 \gamma_1 \end{matrix}\right| & \cdots & \left|\begin{matrix} X \\ \alpha_1 \gamma_x \end{matrix}\right| \\ \left|\begin{matrix} X \\ \alpha_2 \gamma_1 \end{matrix}\right| & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \left|\begin{matrix} X \\ \alpha_x \gamma_1 \end{matrix}\right| & & & \end{vmatrix} \times \left|\begin{matrix} X \\ C \end{matrix}\right|^{\lambda-x} = \Delta_{[x,n]} \cdot \left|\begin{matrix} X \\ C \end{matrix}\right|^{\lambda-x}.$$

Dividing out the common factor from both sides, we have†

$$D_{[\lambda,l]} \cdot \left|\begin{matrix} X \\ C \end{matrix}\right|^{(n-1)_l-\lambda+x} = \Delta_{[x,n]}. \tag{2}$$

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* Metzler, Paper II.

† In a little different notation this is essentially the general relation (3) given in Metzler, Paper II.

Let $\varrho + h = \lambda$, where $\varrho = (r)_l$, and let $\alpha_1, \alpha_2, \cdots, \alpha_\lambda$ be $\alpha_1, \alpha_2, \cdots \alpha_h$, $r_1, r_2, \cdots, r_\varrho$, where $r_1, r_2, \cdots, r_\varrho$ are the combinations of some r numbers taken l at a time. If now we use for $\beta_1, \beta_2, \cdots, \beta_\lambda$ the combinations $\gamma_1, \gamma_2, \cdots, \gamma_h$, $r'_1, r'_2, \cdots, r'_\varrho$, where these are the combinations $(n-l)$ at a time of the n numbers C' consisting of the r numbers in $r_1, r_2, \cdots, r_\varrho$ and $(n-r)$ others, and where r'_i is the complementary of r_i, then we have

$$D_{[\lambda,l]} \cdot D'''_{[\lambda,n-l]} = \begin{vmatrix} \left|\begin{matrix} X \\ \alpha_1\gamma_1 \end{matrix}\right| & \left|\begin{matrix} X \\ \alpha_1\gamma_2 \end{matrix}\right| & \cdots & \left|\begin{matrix} X \\ \alpha_1\gamma_h \end{matrix}\right| \\ \left|\begin{matrix} X \\ \alpha_2\gamma_1 \end{matrix}\right| & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \left|\begin{matrix} X \\ \alpha_h\gamma_1 \end{matrix}\right| & & & \end{vmatrix} \times \left|\begin{matrix} X \\ C' \end{matrix}\right|^\varrho.$$

Dividing out the common factor we have

$$D_{[\lambda,l]} \cdot \left|\begin{matrix} X \\ C' \end{matrix}\right|^{(n-1)_l - \varrho} = \Delta_{[h,n]}. \tag{3}$$

Various other assumptions for the numbers A may be made and corresponding results obtained, some of which will cause the Δ to break up into two or more factors.

Combining (1) and (2) we have

$$\Delta_{[\lambda,n]} \cdot \left|\begin{matrix} X \\ C \end{matrix}\right|^{(n-1)_l - \lambda + x} = \Delta_{[x,n]} \cdot \left|\begin{matrix} X \\ B \end{matrix}\right|^{(n-1)_l}. \tag{4}$$

This may be looked upon as a condensation formula for $\Delta_{[\lambda,n]}$.

If we put $l = n-1$, $x = n-1$, and therefore $\lambda = n$ and $h = 1$, then (1), (2), (4), become*

$$D_{[n,n-1]} \cdot \left|\begin{matrix} X \\ B \end{matrix}\right| = \Delta_{[n,n]}, \tag{1a}$$

$$D_{[n,n-1]} = \Delta_{[n-1,n]}, \tag{2a}$$

$$\Delta_{[n,n]} = \Delta_{[n-1,n]} \cdot \left|\begin{matrix} X \\ B \end{matrix}\right|. \tag{4a}$$

* (2a) is given by Muir, Paper I, and also by Whittaker, loc. cit.; (4a) is given by Muir, Paper II.

In the special case of (3) where $l = n-1$, $r = n-1$ and therefore $\varrho = 1$ and $h = n-1$ we get the same as (2a).

If in $\Delta_{[\lambda,n]}$, α_i, and β_i have no numbers in common, and if α_i and β_j have numbers in common for $\begin{cases} i = 1, 2, \cdots, \lambda-1 \\ j = 1, 2, \cdots, \lambda \end{cases}$ $(j > i)$ then $\Delta_{[\lambda,n]}$ breaks up into factors and is equal to the product of the elements along the principal diagonal. Thus (1) becomes

$$(1b) \qquad D_{[\lambda,l]} \cdot \left|\begin{matrix} X \\ B \end{matrix}\right|^{(n-1)_l} = \left|\begin{matrix} X \\ \alpha_1 \beta_1 \end{matrix}\right| \cdot \left|\begin{matrix} X \\ \alpha_2 \beta_2 \end{matrix}\right| \cdots \left|\begin{matrix} X \\ \alpha_\lambda \beta_\lambda \end{matrix}\right|,$$

(2) becomes

$$(2b) \qquad D_{[\lambda,l]} \cdot \left|\begin{matrix} X \\ C \end{matrix}\right|^{(n-1)_l - \lambda + \varkappa} = \left|\begin{matrix} X \\ \alpha_1 \gamma_1 \end{matrix}\right| \cdot \left|\begin{matrix} X \\ \alpha_2 \gamma_2 \end{matrix}\right| \cdots \left|\begin{matrix} X \\ \alpha_\varkappa \gamma_\varkappa \end{matrix}\right|,$$

(3) becomes*

$$(3b) \qquad D_{[\lambda,l]} \cdot \left|\begin{matrix} X \\ C' \end{matrix}\right|^{(n-1)_l - \varrho} = \left|\begin{matrix} X \\ \alpha_1 \gamma_1 \end{matrix}\right| \cdot \left|\begin{matrix} X \\ \alpha_2 \gamma_2 \end{matrix}\right| \cdots \left|\begin{matrix} X \\ \alpha_h \gamma_h \end{matrix}\right|.$$

If in addition for some value of i α_i and β_i have numbers in common, then the right-hand side of (1b), (2b) and (3b) are zero.

In illustration of the foregoing let $n = 5$, $l = 2$, $h = 1$, and therefore $\lambda = 10$ and $\varkappa = 8$. Let $\alpha_1, \alpha_2, \cdots, \alpha_{10}$ be $x_1 x_2, x_3 x_4, \cdots, x_{19} x_{20}$ and let X be the numbers 1, 2, 3, 4, 5. If we use for B, $x_{16}, x_{17}, x_{18}, x_{19}, x_{20}$, then (2) becomes

$$(2') \qquad D_{[10,2]} \cdot \left|\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ x_{16} & x_{17} & x_{18} & x_{19} & x_{20} \end{matrix}\right|^4$$

$$= \left|\left|\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ x_1 & x_2 & x_{16} & x_{17} & x_{19} \end{matrix}\right| \cdot \left|\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ x_3 & x_4 & x_{16} & x_{17} & x_{20} \end{matrix}\right| \cdots \left|\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ x_{15} & x_{16} & x_{18} & x_{19} & x_{20} \end{matrix}\right|\right|$$

or

$$D_{[10,2]} \cdot \left|\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ x_{16} & x_{17} & x_{18} & x_{19} & x_{20} \end{matrix}\right|^4 = \Delta_{[8,5]}.$$

If we take for $\alpha_1, \alpha_2, \cdots, \alpha_{10}$ the combinations $x_1 x_2$, $x_3 x_4$, $x_5 x_6$, $x_7 x_8$, $\alpha\beta$, $\alpha\gamma$, $\alpha\delta$, $\beta\gamma$, $\beta\delta$, $\gamma\delta$ and, denoting the determinant in this case by $D'_{[10,2]}$ and using α, β, γ, δ, ε for the numbers B, the other conditions remaining the same, then (3) becomes

$$D'_{[10,2]} = \left|\left|\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ x_1 & x_2 & \beta & \gamma & \delta \end{matrix}\right| \cdot \left|\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ x_3 & x_4 & \alpha & \gamma & \delta \end{matrix}\right| \cdot \left|\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ x_5 & x_6 & \alpha & \beta & \delta \end{matrix}\right| \cdot \left|\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ x_7 & x_8 & \alpha & \beta & \gamma \end{matrix}\right|\right|$$

or

$$(3') \qquad D'_{[10,2]} = \Delta_{[4,5]}.$$

* Metzler, Paper I.

If we take for $\alpha_1, \alpha_2, \cdots, \alpha_{10}$ the combinations $\alpha\beta$, $\alpha\gamma$, $\alpha\delta$, $\alpha\varepsilon$, x_1x_2, x_3x_4, $\cdots$, $x_{11}x_{12}$ and denote the determinant in this case by $D''_{[10,2]}$ and let B be $\alpha, \beta, \gamma, \delta, \varepsilon$, then

$$(2'') \qquad D''_{[10,2]} \cdot \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha & \beta & \gamma & \delta & \varepsilon \end{vmatrix}^2 = \begin{vmatrix} \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ x_1 & x_2 & \alpha & \beta & \gamma \end{vmatrix} & \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ x_1 & x_2 & \alpha & \beta & \delta \end{vmatrix} & \cdots & \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ x_1 & x_2 & \alpha & \delta & \varepsilon \end{vmatrix} \\ \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ x_3 & x_4 & \alpha & \beta & \gamma \end{vmatrix} & & & & \\ \vdots & & & & \\ \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ x_{11} & x_{12} & \alpha & \beta & \gamma \end{vmatrix} & & & & \end{vmatrix} = \Delta_{[6,5]}.$$

If instead of using the third compound of $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha & \beta & \gamma & \delta & \varepsilon \end{vmatrix}$ as our multiplier we take the third compound of $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha & x_1 & x_2 & x_3 & x_4 \end{vmatrix}$, we would get, writing but the column numbers since all the row numbers are the same,

$$D''_{[10,2]} \cdot \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha & x_1 & x_2 & x_3 & x_4 \end{vmatrix}^4 = \begin{vmatrix} |\alpha\ \beta\ x_1\ x_2\ x_3| & |\alpha\ \beta\ x_1\ x_2\ x_4| & |\alpha\ \beta\ x_1\ x_3\ x_4| & |\alpha\ \beta\ x_2\ x_3\ x_4| \\ |\alpha\ \gamma\ x_1\ x_2\ x_3| & & & \\ |\alpha\ \delta\ x_1\ x_2\ x_3| & & & \\ |\alpha\ \varepsilon\ x_1\ x_2\ x_3| & & & \end{vmatrix}$$

$$\times \begin{vmatrix} |x_5\ x_6\ \alpha\ x_1\ x_3| & |x_5\ x_6\ \alpha\ x_1\ x_4| & |x_5\ x_6\ \alpha\ x_2\ x_3| & |x_5\ x_6\ \alpha\ x_2\ x_4| \\ |x_7\ x_8\ \alpha\ x_1\ x_3| & & & \\ |x_9\ x_{10}\ \alpha\ x_1\ x_3| & & & \\ |x_{11}\ x_{12}\ \alpha\ x_1\ x_3| & & & \end{vmatrix}$$

$$= \Delta'_{[4,5]} \cdot \Delta_{[4,5]},$$

say. But

$$\Delta'_{[4,5]} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha & \beta & \gamma & \delta & \varepsilon \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha & x_1 & x_2 & x_3 & x_4 \end{vmatrix}^3$$

since it is an example of the extensional of the theorem of Reiss or Picquet.* We have therefore,

$$(2''') \qquad D''_{[10,2]} \cdot \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha & x_1 & x_2 & x_3 & x_4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha & \beta & \gamma & \delta & \varepsilon \end{vmatrix} \cdot \Delta_{[4,5]}.$$

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* Picquet, Journ. de l'Ec. Polyt., cah. 45, pp. 201–243. Reiss, Beiträge zur Theorie der Determinanten.

Since writing this paper my attention has been called to some results given by Reiss. On pages 64–66 besides giving some special cases of (2) when n and l differ by unity he gives a few examples where n and l differ by two.

From (2′) and (2‴) it follows that

$$(4') \qquad \Delta_{[6,5]} \cdot \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha & x_1 & x_2 & x_3 & x_4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha & \beta & \gamma & \delta & \varepsilon \end{vmatrix}^3 \Delta_{[4,5]}.$$

If for $\alpha_1, \alpha_2 \cdots \alpha_{10}$ we take $pq, r\beta, s\gamma, t\delta, \alpha\beta, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta, \gamma\delta$ and denote the determinant by $D'''_{[10,2]}$, then, using as multiplier the third compound of $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha & \beta & \gamma & \delta & \varepsilon \end{vmatrix}$, we get*

$$(3'_b) \qquad D'''_{[10,2]} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ p & q & \beta & \gamma & \delta \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ r & \alpha & \beta & \gamma & \delta \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ s & \alpha & \beta & \gamma & \delta \end{vmatrix} \cdot \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ t & \alpha & \beta & \gamma & \delta \end{vmatrix}.$$

Here p, q, r, s, t may be any numbers as long as

(1) p or $q \neq \beta$ or γ or δ,

(2) r or s or $t \neq \alpha$ or β or γ or δ.

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* Metzler, Paper I.

NEW YORK STATE COLLEGE FOR TEACHERS,
March 1924.

A CONTRIBUTION TO THE THEORY OF INTERPOLATION.

By Norbert Wiener.

The familiar formulae of trigonometric and polynomial interpolation may be put in the form of the solutions of minimization-problems—we are required to find the trigonometric polynomial of lowest order, or the algebraic polynomial of lowest degree, assuming assigned values for an assigned set of values of the independent variable. In these problems the quantity to be minimized varies through integral values only. It is hence difficult or impossible to apply methods resembling those of the calculus of variations. In the present paper we shall consider a mode of interpolation where the interpolating function is required to satisfy a certain minimum condition in which the quantity to be minimized varies continuously, and in which the methods employed will be similar to those ordinarily employed in the calculus of variations.

The particular problem which we shall discuss is the following. Let C be a closed curve in the plane, containing in its interior the points $P_1, P_2, \cdots, P_n$. We are required to determine that harmonic function $f(P)$ assuming at $P_1, \cdots, P_n$ the given values $F_1, \cdots, F_n$, with no singularities within C, assuming continuously on C boundary values $F(P)$, and such that $\int_C \{F(P)\}^2 ds$ is a minimum. It is possible to develop closely analogous interpolation problems dealing with functions of a complex variable instead of harmonic functions, and the treatment of these will differ in no essential respect from that given here. It is to be noted that on a curve passing through $P_1, \cdots, P_n$ our harmonic function $f(P)$ defines an interpolating function of a single variable between $F_1, \cdots, F_n$.

Our problem resolves itself into two parts: the formal determination of the solution of the interpolation problem, and the necessary existence proofs. If we apply the familiar methods of the calculus of variations, we see that it is at any rate a necessary condition for the minimization of $\int_C [F(P)]^2 ds$ by $F(P)$ that, if $H(P)$ is a continuous function on C determining within C the harmonic function $h(P)$ which vanishes at $P_1, \cdots, P_n$, then

$$\int_C [F(P)+uH(P)]^2 ds = \int_C [F(P)]^2 ds + 2u\int_C F(P)H(P)ds + u^2\int_C [H(P)]^2 ds$$

assumes its minimum value when $u = 0$. It results at once from this that

$$\int_C F(P)H(P)ds = 0.$$

Now let $G(P, Q)$ be the Green's function of C. If C consists in a single closed analytic curve*, $\partial G(P, P_k)/\partial n$ will be continuous, and

$$\int_C H(P) \frac{\partial G(P, P_k)}{\partial n} ds = h(P_k) = 0,$$

that is, $F(P)$ is orthogonal to all the continuous functions orthogonal to every $\partial G(P, P_k)/\partial n$, and will hence be linearly dependent on these latter functions, for it will be possible to orthogonalize the functions $\partial G(P, P_k)/\partial n$ and to give $F(P)$ a formal development in terms of this orthogonal set, and the difference between F and this development will be orthogonal at once to all the functions $\partial G(P, P_k)/\partial n$ and to all the functions orthogonal to all these functions. Being thus self-orthogonal, it will vanish.

If $F(P)$ is of the form $\sum_1^n a_k \partial G(P, P_k)/\partial n$, we shall have

$$f(Q) = \sum_1^n \frac{a_k}{2\pi} \int_C \frac{\partial G(P, P_k)}{\partial n} \frac{\partial G(P, Q)}{\partial n} ds.$$

Our interpolation problem is then formally solved by the determination of the coefficients a_k in such a manner that

$$\sum_1^n \frac{a_k}{2\pi} \int_C \frac{\partial G(P, P_k)}{\partial n} \frac{\partial G(P, P_j)}{\partial n} ds = F_j.$$

Let us write A_{ij} for $\int_C \frac{\partial G(P, P_i)}{\partial n} \frac{\partial G(P, P_j)}{\partial n} ds$. Then the determination of the coefficients a_k is possible in one and only one way unless

$$\begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{vmatrix} = 0.$$

The vanishing of this determinant signifies, however, the existence of a relation of linear dependence† among the quantities $\partial G(P, P_k)/\partial n$. If this relation holds, there will be a set of non-zero b_k's such that, for any function $h(P)$ continuous on C,

$$\sum_1^n \frac{b_k}{2\pi} \int_C h(P) \frac{\partial G(P, P_k)}{\partial n} ds = 0;$$

* Cf. Osgood, Funktionentheorie, III, 13, § 6, Satz 3.

† Cf. Mrs. A. J. Pell, Biorthogonal systems of functions, Trans. Amer. Math. Soc., vol. 12 (1911), p. 157.

that is, any harmonic function continuous on and within C and vanishing at $P_1, \cdots, P_{n-1}$ will also vanish at P_n. This is however impossible, for the harmonic function

$$R\left\{\frac{(P-P_1)(P-P_2)\cdots(P-P_{n-1})}{(P_n-P_1)(P_n-P_2)\cdots(P_n-P_{n-1})}\right\}$$

will have no singularities in the finite part of the plane, will vanish at $P_1, \cdots, P_{n-1}$, and will equal 1 at P_n. Hence our determinant cannot vanish and the coefficients a_k can be determined in one way and one way only.

Let us then determine our a_k's by the condition that $\sum_1^n A_{ij}\, a_i = 2\pi F_j$, and let us put

$$F(P) = \sum_1^n a_k \frac{\partial G(P, P_k)}{\partial n}.$$

Let $e(P)$ be any function harmonic within C, assuming the same values as $f(P)$ at $P_1, \cdots, P_n$, bounded, and assuming the summable boundary values $E(P)$ on C. Then

$$\begin{aligned}
\int_C [E(P)]^2\, ds &= \int_C \{F(P) + [E(P) - F(P)]\}^2\, ds \\
&= \int_C [F(P)]^2\, ds + \int_C [E(P) - F(P)]^2\, ds \\
&\qquad + 2\int_C F(P)\,[E(P) - F(P)]\, ds \\
&= \int_C [F(P)]^2\, ds + \int_C [E(P) - F(P)]^2\, ds \\
&\qquad + 2\sum_1^n a_k \int_C \frac{\partial G(P, P_k)}{\partial n}\,[E(P) - F(P)]\, ds \\
&= \int_C [F(P)]^2\, ds + \int_C [E(P) - F(P)]^2\, ds \\
&\qquad + 2\sum_1^n a_k\,[e(P_k) - f(P_k)] \\
&= \int_C [F(P)]^2\, ds + \int_C [E(P) - F(P)]^2\, ds.
\end{aligned}$$

This is clearly a true minimum when $E(P) = F(P)$.

Let R be any closed set of points interior to C. Let C be of such a character that for any positive θ, if $\overline{PQ} \geqq \theta$, $\partial G(P, Q)/\partial n$ is a continuous function of its two arguments. It may readily be proved that this is the case when C consists of a single closed analytic curve, for then $G(P, Q)$ will be analytic even if one argument lies on the boundary, provided P is not in the neighborhood of Q. Then if Q lies on R and P on C, $\partial G/\partial n$

will be uniformly continuous. It follows that, given any ϵ, there is an η such that if P lies on C, and Q and Q' on R, while $\overline{QQ'} < \eta$, then

$$\left| \frac{\partial G(P,Q)}{\partial n} - \frac{\partial G(P,Q')}{\partial n} \right| < \epsilon.$$

Now let $\Phi(P)$ be any function defined over C, summable, and of summable square. The harmonic function $\varphi(Q)$ which it will define over the interior of C will be

$$\frac{1}{2\pi} \int_C \Phi(P) \frac{\partial G(P,Q)}{\partial n} \, ds.$$

Then by the Schwarz inequality, if $\overline{QQ'} < \eta$

$$\begin{aligned} |\varphi(Q) - \varphi(Q')| &= \frac{1}{2\pi} \int_C \Phi(P) \left[\frac{\partial G(P,Q)}{\partial n} - \frac{\partial G(P,Q')}{\partial n} \right] ds \\ &\leqq \frac{\epsilon}{2\pi} \sqrt{\int_C [\Phi(P)]^2 \, ds}. \end{aligned}$$

It immediately follows that the class of all functions harmonic within C and corresponding to boundary-values $\Phi(P)$ on C that are summable, of summable square, and such that $\int_C [\Phi(P)]^2 \, ds \leqq 1$, is an equicontinuous class over R. It is manifestly uniformly bounded. Hence any sequence of such functions will contain a uniformly convergent sub-sequence.

Now let R be defined as above, and let $P_1, P_2, \cdots$ be an infinite set of points within R. Let $\Phi(P)$ be a summable function of summable square defined on C, defining over the interior of C the harmonic function $\varphi(Q)$. Let $\varphi_n(Q)$ be the interpolation harmonic function given by our method and corresponding to the given values $\varphi(P_k)$ at P_k $(k = 1, \cdots, n)$. Let $\Phi_n(P)$ be the boundary values corresponding to $\varphi_n(Q)$. Clearly

$$\int_C [\Phi_n(P)]^2 \, ds \leqq \int_C [\Phi(P)]^2 \, ds.$$

Hence it is possible to pick out from the sequence $\varphi_n(P)$ a sub-sequence converging uniformly. It can, however, converge uniformly only to a function assuming at $P_1, \cdots$ the same values as φ. If there is only one such function, as will be the case if the P_k's lie everywhere dense along a closed contour, the sequence will necessarily converge to φ itself.

In general, if the P_k's lie everywhere dense on a closed set of points S in R the entire sequence of the φ_n's will converge uniformly on S. For let ϵ be any positive number, let η be determined in terms of ϵ as above, and let n be so large that no point of S is at a distance greater than η

from some one of the points $P_1, \cdots, P_n$. Let Q be a point of S, and let it be at a distance not greater than η from P_k. Then

$$\begin{aligned} |\varphi(Q)-\varphi_n(Q)| &\leqq |\varphi(Q)-\varphi(P_k)|+|\varphi_n(Q)-\varphi_n(P_k)|+|\varphi(P_k)-\varphi_n(P_k)| \\ &= |\varphi(Q)-\varphi(P_k)|+|\varphi_n(Q)-\varphi_n(P_k)| \\ &\leqq \frac{\varepsilon}{\pi}\sqrt{\int_C [\Phi(P)]^2\,ds}. \end{aligned}$$

From this it follows at once that φ_n converges uniformly to φ over S. C may be, for instance a circle and S, a smaller circle concentric with C, or a segment of the diameter of C. We have then determined a process of interpolation which will apply and converge, under the conditions we have here developed, not only in the case of a two-dimensional harmonic function, but in the case of a function of a single variable.

Let C be a circle of radius r. Let us set up polar coordinates with the center of C as pole. Let $P_1, \cdots, P_n$ lie on the initial radius, with coordinates $(\varrho_1, 0), \cdots, (\varrho_n, 0)$. Let Q be the point $(\varrho, 0)$. Then it may be verified by direct computation that

$$\frac{1}{2\pi}\int_C \frac{\partial G(P, P_k)}{\partial n}\,\frac{\partial G(P, Q)}{\partial n}\,ds = \frac{r+\varrho\varrho_k}{r(r-\varrho\varrho_k)}$$

It will follow that our interpolating function for assigned values at $P_1, \cdots, P_n$ will assume over the initial radius the form

$$I(\varrho) = \frac{P(\varrho)}{\prod_1^n \left(1-\frac{\varrho\varrho_k}{r}\right)}$$

where $P(\varrho)$ is a polynomial of the nth degree in ϱ, and

$$\lim_{r\to\infty} \prod_1^n \left(1-\frac{\varrho\varrho_k}{r}\right) = 1.$$

If r is sufficiently large, $|I(\varrho)-P(\varrho)|<\varepsilon$ over a given bounded segment of the initial radius. That is, $P(\varrho)$ is the Lagrange interpolation polynomial for a function differing from $I(\varrho)$ at $P_1, \cdots, P_n$ by less than ε. As the Lagrange interpolation polynomial is a continuous function of the values between which interpolation is made, as r becomes infinite, $I(\varrho)$ approaches the Lagrange interpolation polynomial corresponding to the given values $I(\varrho_k)$ at $P_1, \cdots, P_n$. Hence $I(\varrho)$ itself approaches the Lagrange interpolation polynomial.

A PROPERTY OF CYCLOTOMIC INTEGERS AND ITS RELATION TO FERMAT'S LAST THEOREM.*

(SECOND PAPER.)

By H. S. Vandiver.

If

$$x^p + y^p + z^p = 0 \tag{1}$$

is satisfied in integers prime to each other and to the odd prime p, and $\alpha = e^{2\pi i/p}$ then the writer† has shown that

$$\prod_{r=1}^{k-1} \prod_{r=1}^{\left[\frac{rp}{k}\right]} (x + \alpha^{[1:r]} y)^2 = \alpha^{\frac{-2kyq(k)}{x+y}} \omega^{2p} \tag{2}$$

where k is an integer, $1 < k < p$, ω is an integer in the field $\Omega(\alpha)$,

$$q(k) = \frac{k^{p-1} - 1}{p}$$

and $[1:r]$ is the least positive solution of $Xr \equiv 1 \pmod{p}$. From this relation the Kummer criteria

$$B_n f_{p-2n}(t) \equiv 0 \pmod{p}, \qquad (n = 1, 2, \cdots, (p-3)/2),$$

were derived, the B's being the numbers of Bernoulli, $B_1 = 1/6$, $B_2 = 1/30$, etc.

$$f_i(t) = t + 2^{i-1} t^2 + \cdots + (p-1)^{i-1} t^{p-1}$$

$(-t)$ being any of the ratios

$$\frac{x}{y}, \quad \frac{y}{x}, \quad \frac{x}{z}, \quad \frac{z}{x}, \quad \frac{y}{z}, \quad \frac{z}{y}.$$

By means of the Kummer criteria, Mirimanoff‡ showed that if (1) is satisfied, $xyz \not\equiv 0 \pmod{p}$, then $B_{\mu-1} = B_{\mu-2} = B_{\mu-3} = B_{\mu-4} \equiv 0 \pmod{p}$, $\mu = (p-1)/2$.

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* Presented to the American Mathematical Society April 24th, 1915 and April 27th, 1918.

† Ann. of Math., 2nd ser., vol. 21 (1919), p. 78.

‡ Crelle, vol. 128 (1905), pp. 45–68. Cf. also, Dickson's History of Theory of Numbers, vol. 2, chapter 26, refs. 57, 64, 76.

In the present paper, extensions of the Kummer criteria will be obtained by the use of (2) and the conditions $B_s \equiv 0 \pmod{p^2}$, $s = (lp+1)/2$, $l = p-4, p-6, p-8, p-10$, derived therefrom. For the modulus p, these congruences reduce to the above relations of Mirimanoff.*

Although the results in this paper have been known to me for the past eight years, I was enabled to write the paper in final form during the present year through a grant relieving me of the duties of teaching, from the Heckscher Foundation for the Advancement of Research, established by August Heckscher, at Cornell University.

1. It will first be shown that

$$(x+y)^p \equiv x+y \pmod{p^3}$$

if (1) is satisfied in integers, $xyz \not\equiv 0 \pmod{p}$. In this case it is known that

$$x+y = \nu^p \tag{3}$$

where ν is an integer. It is also known† that

$$\frac{x^p+y^p}{x+y} \equiv 1 \pmod{p^3},$$

which gives $\nu^p \equiv -z^p \pmod{p^3}$, and therefore $\nu \equiv -z \pmod{p^2}$, whence

$$(x+y)^p \equiv \nu^p \pmod{p^3}$$

and (3) gives

$$(x+y)^p \equiv (x+y) \pmod{p^3}$$

or

$$(x+y)^{p-1} \equiv 1 \pmod{p^3}. \tag{4}$$

2. The relation (2) may be written, after raising to the power $(p-1)/2$,

$$\alpha^{\frac{kyq(k)}{x+y}} \prod_r (x+\alpha^{[1:r]}y)^{p-1} = \omega^{p(p-1)}.$$

We have

$$\omega = a+(1-\alpha)^s\beta$$

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* This reduction is effected by using a relation of Kummer's, Crelle, vol. 41 (1851), p. 371, line 9.

† Vandiver, Transactions Amer. Math. Soc., vol. 15 (1914), p. 203.

where β is an algebraic integer in $\Omega(\alpha)$, and a is a rational integer prime to p, $s>0$. Hence $\omega^{p-1} = a^{p-1} + (1-\alpha)^s \beta_1$, β_1 an algebraic integer in $\Omega(\alpha)$. Also $a^{p-1} \equiv 1 \pmod{p}$ and, since $p = (1-\alpha)^{p-1}\varepsilon$, where ε is a unit in $\Omega(\alpha)$, then

$$\omega^{p-1} = 1 + (1-\alpha)^s \beta_1 + (1-\alpha)^{p-1} \beta_2,$$

β_2 being an integer in $\Omega(\alpha)$. Write $\theta(\alpha)$ for ω^{p-1}, then $\theta(x) = 1$ for $x = 1$. We may then write

$$\alpha^{\frac{kyq(k)}{x+y}} \prod_r (x + \alpha^{[1:r]} y)^{p-1} = (\theta(\alpha))^p.$$

This shows that the rational integral function S of A

$$S = A^{\frac{kyq(k)}{x+y}} \prod_r (x + A^{[1:r]} y)^{p-1} - (\theta(A))^p$$

vanishes if A is equal to any root of $(u^p-1)/(u-1) = 0$. Since the latter equation is irreducible in the rational field, we have

$$S = W \cdot \frac{A^p - 1}{A - 1},$$

where W is a rational integral function of A with rational integral coefficients. Set $A = e^v$ where e is the Napierian base. Then

$$e^{\frac{vkyq(k)}{x+y}} \prod_r (x + e^{v[1:r]} y)^{p-1} = (\theta(e^v))^p + W \cdot \frac{e^{vp} - 1}{e^v - 1}. \tag{5}$$

Consider the expression

$$W \cdot \frac{e^{vp} - 1}{e^v - 1}.$$

Divide W by $e^v - 1$, considering W as a polynomial in e^v. The whole expression may then be written

$$(e^{vp} - 1) V + c \frac{e^{vp} - 1}{e^v - 1},$$

where c is a rational integer, and V is a rational integral function of e^v with coefficients rational integers. Substituting this expression in (5) and setting $v = 0$, we obtain

$$(x+y)^{(k-1)(p-1)^s/2} = 1 + cp, \tag{6}$$

since $\theta(1) = 1$. Using (4) and (6) we have $c = c_1 p^2$, where c_1 is a rational integer. We may therefore put (5) in the form

$$(7)\quad e^{\frac{vkyq(k)}{x+y}} \prod_r (x + e^{v[1:r]}y)^{p-1} = (\theta(e^v))^p + V(e^{vp}-1) + c_1 p^2 \frac{e^{vp}-1}{e^v-1}.$$

In this expression take logarithms of both sides and differentiate np times with respect to v; the result takes the form (if W is the right hand member of (7))

$$(7a)\quad (p-1)\sum_{r=1}^{k-1}\sum_{r=1}^{\left[\frac{rp}{k}\right]} D_{np}(\log(x + e^{v[1:r]}y)) = D_{np-1}\left(\frac{D(W)}{W}\right)$$
$$= W^{-1}D_{np}(W) + (np-1)D_{np-1}(W)D(W^{-1}) + \cdots + D(W)D_{np-1}(W^{-1})$$

where $D_s(X)$ designates the sth derivative of X with respect to v, and $D(X) = D_1(X)$. If we substitute $v = 0$ in this relation we obtain on the right hand side the sum of a number of fractions each of whose denominators is of the form $1 + t\,p^2$, (since if $W(e^v) = W$ then $W(1) \equiv 1 \pmod{p^2}$) and each numerator divisible by p^2, except the term $[W^{-1}(1)\,D_{np}(W)]_{v=0}$. We then write $D_{np} = [D_{np}(\log(x + e^v y))]_{v=0}$ and (7a) gives, if $\sum [1:r]^s$ stands for

$$\sum_{r=1}^{k-1}\sum_{r=1}^{\left[\frac{rp}{k}\right]} [1:r]^s,$$

$$(7b)\quad \begin{aligned}(p-1)\,D_{np}\sum[1:r]^{np} &\equiv [D_{np}(\theta^p + (e^{vp}-1)V)]_{v=0} \pmod{p^2}\\ &\equiv [D_{np}(\theta^p)]_{v=0} + [D_{np}((e^{vp}-1)V)]_{v=0} \pmod{p^2}.\end{aligned}$$

Now

$$[D_{np}((e^{vp}-1)V)]_{v=0}$$
$$\equiv [(e^{vp}-1)D_{np}(V) + np\,D_{np-1}(V)\,D(e^{vp}-1)]_{v=0} \pmod{p^2}$$

since each derivative of $e^{vp}-1$ with respect to v after the first is, for $v = 0$, divisible by p^2. Reducing the right hand member we see that it is also divisible by p^2, and we therefore have from (7b)

$$(p-1)\,D_{np}\sum[1:r]^{np} \equiv p\,[D_{np-1}(\theta^{p-1}D(\theta))]_{v=0} \pmod{p^2}.$$

Now

$$\theta^{p-1}D(\theta) = \theta^p[D(\theta)\theta^{-1}],$$

and differentiating the right hand member regarded as a product, we have

$$(8)\quad \begin{aligned}[D_s(\theta^{p-1}D(\theta))]_{v=0} &\equiv [D_s\,\theta^{-1}D(\theta)]_{v=0} \pmod{p}\\ &\equiv [D_{s+1}(\log\theta)]_{v=0} \pmod{p}\end{aligned}$$

and we may then write

$$(8a) \qquad (p-1)D_{np}\sum[1:r]^{np} \equiv p[D_{np}(\log\theta)]_{v=0} \pmod{p^2}.$$

Set

$$x+\alpha y = (\mathfrak{P}^{(1)}\mathfrak{P}^{(2)}\cdots\mathfrak{P}^{(i)})^p$$

where the $\mathfrak{P}$'s are ideal primes not necessarily distinct. Since each factor of $(x^p+y^p)/(x+y)$ is of the form $pf+1$, then the $\mathfrak{P}$'s are all ideals of the first degree. We may then set $N(\mathfrak{P}^{(s)}) = q_s$ where q is a rational prime and N stands for the norm. Denote by $\mathfrak{P}_a$ the ideal obtained from $\mathfrak{P}$ by the substitution (α/α^a). Evidently

$$\prod_r((x+\alpha^{[1:r]}y)) = \left(\prod_r\prod_{s=1}^{i}\mathfrak{P}^{(s)}_{[1:r]}\right)^p.$$

In my Annals paper* it is shown that $\prod_r\prod_{s=1}^{i}\mathfrak{P}^{(s)}_{[1:r]}$ is the principal ideal (ω) in (2) of the present paper. Also the way this result is derived shows that

$$(\omega(\alpha^{-1})) = \left(\prod_{s=1}^{i}\prod_{u=1}^{k-1}\psi^{(s)}_u(\alpha)\right)$$

$$(\psi^{(s)}_u(\alpha)) = \prod_w\prod_t\mathfrak{P}_{[1:t]}$$

where t ranges over the integers which satisfy

$$\frac{wp}{u+1} < t < \frac{wp}{u}$$

and w ranges over the set $1, 2, \cdots, u$.

The function $\psi^{(s)}_u(\alpha)$ is a Jacobi cyclotomic number† and

$$\psi^{(s)}_u(\alpha) = \sum_h \alpha^{-(u+1)h+\text{ind.}(g^h+1)}$$

where h ranges over the integers $0, 1, \cdots, q_s-2$, excepting $(q_s-1)/2$, ind. x being defined as i in the relation $y^i \equiv x \pmod{q_s}$, g is a primitive root of q_s such that the ideal $(g^{(q_s-1)/p}-\alpha, q_s)$ equals $\mathfrak{P}_s$. We have

$$\psi_s(\alpha)\,\psi_s(\alpha^{-1}) = q_s.$$

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* L. c. pp. 74–75.

† Report of the committee on Algebraic Numbers, Bull. Nat. Research Council, Feb. 1923, pp. 50–53.

Hence

$$\omega(\alpha^{-1}) = E\prod_{s=1}^{i}\prod_{u=1}^{k-1}\psi_u^{(s)}(\alpha) \tag{9}$$

where E is a unit in $\Omega(\alpha)$. By a known theorem $E = \alpha^c\varepsilon$ where ε is a unit belonging to the field $\Omega(\alpha+\alpha^{-1})$. We then have by (2)

$$(\omega(\alpha)\,\omega(\alpha^{-1}))^p = \left(\frac{x^p+y^p}{x+y}\right)^{(k-1)} = \varepsilon^{2p}\prod_{s=1}^{i} q_s^{(k-1)p},$$

whence $\varepsilon^{2p} = 1$, $\varepsilon = \pm 1$. Then from (9)

$$\theta = \omega^{p-1} = \alpha^{c(p-1)}\prod_{s=1}^{i}\prod_{u=1}^{k-1}\psi_u^{(s)}(\alpha^{-1})^{p-1},$$

and therefore

$$\theta(e^v) = e^{vc(p-1)}\prod_{s=1}^{i}\prod_{u=1}^{k-1}\psi_u^{(s)}(e^{-v})^{p-1} + Y\frac{e^{vh}-1}{e^v-1}$$

where Y is an integral function of Y with rational integral coefficients. Put the term involving the ψ's equal to F, then

$$\frac{\theta}{F} = 1+\frac{Y}{F}\,\frac{e^{vh}-1}{e^v-1}.$$

If we take logarithms and differentiate both members, the right hand side may be reduced to the sum of a number of simple fractions all of whose denominators are of the form

$$F^a\left(1+\frac{Y}{F}\,\frac{e^{vp}-1}{e^v-1}\right)^b$$

and all of whose numerators are integral functions of e^v with integral coefficients. Also, after zero is substituted for v we find that each denominator is prime to p since $[F]_{v=0}$ is prime to p since ω is prime to it, and

$$\left[1+\frac{Y}{F}\,\frac{e^{vp}-1}{e^v-1}\right]_{v=0} \equiv 1 \pmod{p},$$

each numerator being divisible by p. We may then write

$$[D_{np}(\log\theta)]_{v=0} \equiv (p-1)\,[D_{np}(\sum_{s}\sum_{u}\log\psi_u^{(s)}(e^{-v}))]_{v=0} \pmod{p}$$

and from (8a)

$$(10)\qquad (p-1)\,D_{np}\sum[1:r]^{np} \equiv (-1)^{np}\,p(p-1)\,D_{np}(\sum_{s}\sum_{u}\log\psi_u^{(s)}(e^{v})) \pmod{p^2}.$$

We have also

$$[D_{np}\,\psi]_{v=0} \equiv [D_n\,\psi]_{v=0} \pmod{p}.$$

For it is easily seen that

$$[D_r(\psi^{p-1}D\psi)]_{v=0} \equiv [\psi^p D_r(\psi^{-1}D\psi)]_{v=0} \pmod{p},$$

and if

$$\psi^{p-1}D\psi = a_0 + a_1 e^{v} + a_2 e^{2v} + \cdots + a_s e^{sv}$$

then

$$D_{np-1}(\psi^{p-1}D\psi) = a_1 e^{v} + 2^{np-1}a_2 e^{2v} + \cdots + s^{np-1}a_s e^{sv}.$$

Hence for $n>1$ we have by Fermat's theorem

$$D_{n-1}(\psi^{p-1}D\psi) \equiv D_{np-1}(\psi^{p-1}D\psi) \pmod{p}$$

$$[D_{np-1}(\psi^{-1}D)]_{v=0} \equiv [D_{n-1}(\psi^{-1}D)]_{v=0} \pmod{p}$$

$$(11)\qquad [D_{np}\log\psi]_{v=0} \equiv [D_n\log\psi]_{v=0} \pmod{p}, \quad \text{for } n>1.$$

In a similar way we have

$$[D_r\log\psi(e^{-v})]_{v=0} \equiv (-1)^r\,[D_r\log\psi(e^{v})]_{v=0} \pmod{p}.$$

3. If ζ is an integer in $\Omega(\alpha)$, $\mathfrak{P}$ is an ideal prime in Ω which is prime to ζ and p then we set*

$$\zeta^{\frac{N(\mathfrak{P})-1}{p}} \equiv \left\{\frac{\zeta}{\mathfrak{P}}\right\} \pmod{\mathfrak{P}}.$$

* Hilbert, Ber. der Deutsch. Math. Ver., 4 (1894), p. 365.

It follows that ζ is a perfect pth power in Ω modulus $\mathfrak{P}$, if and only if

$$\left\{\frac{\zeta}{\mathfrak{P}}\right\} = 1.$$

Now Kummer* has proved that, retaining the former notation as to $\mathfrak{P}$ and ψ,

$$\left\{\frac{E_m(\alpha)}{\mathfrak{P}^{(s)}}\right\} = \alpha^M,$$

where

$$M \equiv \frac{\gamma^{2m}-1}{2(1+j^{p-2m}-(j+1)^{p-2m})}[D_{p-2m}(\log \psi_j^{(s)})]_{v=0} \pmod{p}$$

and

$$E_m(\alpha) = e(\alpha)\, e(\alpha^{\gamma})^{\gamma^{-2m}}\, e(\alpha^{\gamma^2})^{\gamma^{-4m}} \cdots (e^{\gamma^{\mu-1}})^{\gamma^{-2m(\mu-1)}}$$

γ being a primitive root of p, $\mu = (p-1)/2$ and j any integer, $0 < j < p-1$. We then have

$$\frac{2M(1+j^{p-2m}-(j+1)^{p-2m})}{\gamma^{2m}-1} \equiv [D_{p-2m} \log \psi_j^{(s)}]_{v=0} \pmod{p}.$$

Setting $j = 1, 2, \cdots, k-1$, in turn and adding we obtain

$$\frac{2M(k-k^{p-2m})}{\gamma^{2m}-1} \equiv \sum_{j=1}^{k-1} [D_{p-2m} \log \psi_j^{(s)}]_{v=0} \pmod{p}$$

We also have

$$\left\{-\frac{E_m(\alpha)}{\prod_{s=1}^{i} \mathfrak{P}^{(s)}}\right\} = \alpha^N$$

where

$$N = \text{ind. } E_m(\alpha) \equiv \frac{\gamma^{2m}-1}{2(k-k^{p-2m})} \sum_{s=1}^{i} \sum_{j=1}^{k-1} [D_{p-2m} \log \psi_j^{(s)}]_{v=0} \pmod{p}.$$

From (10) and (11) we have for $n = p-2m$

$$(12) \quad D_{p(p-2m)} \sum [1 : r]^{p(p-2m)} \equiv -p \frac{2 \text{ ind. } E_m(\alpha)\,(k-k^{p-2m})}{\gamma^{2m}-1} \pmod{p^2}.$$

* Crelle, vol. 44 (1852), p. 103.

Now by definition we have, if β is some rational integer,

$$\gamma^{p(p-2m)}\,[1:\gamma]^{p(p-2m)} = (1+\beta p)^{p(p-2m)} \equiv 1 \pmod{p^2},$$

and therefore

$$[1:\gamma]^{p(p-2m)} \equiv \frac{1}{\gamma^{p(p-2m)}} \pmod{p^2}$$

or

$$\sum_{\gamma}[1:\gamma]^{p(p-2m)} \equiv \sum_{\gamma}\gamma^{(2m-1)p} \pmod{p^2}.$$

The writer has proved* that

$$\frac{(n^i-1)\sum_{a=1}^{p-1}a^i}{p} = \sum_{a=1}^{p-1}\sum_{s=1}^{i}\binom{i}{s}\left(\frac{y_a}{a}\right)^s p^{s-1},$$

where i is an arbitrary integer, n is an integer prime to p and not unity, and

$$y_a \equiv -\frac{a}{p} \pmod{n}, \qquad (0 \leqq y_a < n).$$

Setting $i = l = (2m-1)p+1$ and $n = k$, this gives, modulo p^2,

$$\frac{(k^l-1)\sum_{a=1}^{p-1}a^l}{pl} \equiv \sum_{a=1}^{p-1} y_a\, a^{l-1} \pmod{p^2}. \tag{13}$$

We shall now prove that

$$\sum_{a=1}^{p-1} a^l \equiv (-1)^{l/2-1}\, p\, B_{l/2} \pmod{p^3}, \quad (p>3). \tag{14}$$

The Bernoulli summation formula gives

$$\sum_{a=1}^{p-1} a^l \equiv p\,h^l + p\,\frac{l}{2}(ph)^{l-1} + p^2\,\frac{l(l-1)}{2\cdot 3}(ph)^{l-2} \pmod{p^3}.$$

Where $h_1 = \frac{1}{2}$, $h^{2i} = (-1)^{i-1}B_i$, $h^{2i+1} = 0$, $(i>0)$.

This congruence follows from the fact that, because of the von Staudt-Clausen Theorem, ph^r is a fraction with denominator not divisible by p for

* Ann. of Math., 2nd ser., vol. 18 (1917), p. 112, relation (7a).

any r. The third term on the right is divisible by p^3 because $l-1 \equiv 0 \pmod{p}$ and we may assume $p \neq 3$. Further $h^{l-1} = 0$ and the relation (14) is proved. Substituting in (13) we have

$$\text{(14a)} \qquad \frac{(-1)^{l/2-1} B_{l/2}(k^l-1)}{l} \equiv \sum_{a=1}^{p-1} y_a a^{l-1} \pmod{p^2}.$$

Now applying the process to the right hand member which is given in the author's Annals paper already cited, pages 113 and 114, § 7, we obtain

$$\text{(14b)} \qquad \frac{(-1)^{l/2} B_{l/2}(k^l-1)}{k^{l-1} l} \equiv \sum_r r^{l-1} \pmod{p^2}$$

where $\sum_r$ is defined as before. The relation (12) then gives, after division by p,

$$\frac{(-1)^{l/2+1} B_{l/2} D_{p(p-2m)}(k^l-1)}{k^{l-1} lp} \equiv \frac{2(k-k^{p-2m})}{\gamma^{2m}-1} \text{ ind. } E_m(\alpha) \pmod{p}.$$

We may assume that k is a primitive root of p and therefore $kl \not\equiv 1 \pmod{p}$, and also $B_{l/2} D_{p(p-2m)} \equiv 0 \pmod{p}$. We further have

$$\frac{k^l-1}{k^{l-1}} \equiv k-k^{p-2m} \pmod{p},$$

which gives

$$\frac{(-1)^{l/2+1} B_{l/2} D_{p(p-2m)}(\gamma^{2m}-1)}{2p} \equiv \text{ ind. } E_m(\alpha) \pmod{p}.$$

Using this in connection with the formula*

$$\text{ind.}\left(\frac{1-\alpha^s}{1-\alpha}\right) \equiv \frac{1}{2}(s-1) \text{ ind. } \alpha - 2\sum_{m=1}^{\mu-1} \frac{\text{ind. } E_m(\alpha^s) - \text{ind. } E_m(\alpha)}{\gamma^{2m}-1} \pmod{p}$$

$$\mu = (p-1)/2,$$

we obtain, if $\prod \mathfrak{P}^{(i)} = Q$,

$$\text{(15)} \quad \text{ind.}\left(\frac{1-\alpha^s}{1-\alpha}\right) \equiv \frac{N(Q)-1}{2p}(s-1) + \sum_{m=1}^{\mu-1} \frac{(-1)^{l/2} B_{l/2} D_{p(p-2m)}(s^{2m}-1)}{2mp} \pmod{p}.$$

* Kummer, Crelle, vol. 56, p. 277.

4. In the relation (2) put $k = 2$, and α^h for α. For $k = 2$, r ranges over the integers $1, 2, \cdots, \mu$. If in (2) there is an r such that $h[1:r] \equiv 1 \pmod{p}$, then set α^{-h} for α^h throughout the relation (2). Then we cannot have any exponent of α which is congruent to 1 (mod. p), for if such were the case we have $-h[1:r_1] \equiv 1 \pmod{p}$, where r_1 is one of the r's, and this gives $[1:r] \equiv -[1:r_1] \pmod{p}$ which is impossible. Equating pth power characters of each side of the new (2) with respect to Q, we obtain, since each binomial factor is prime to $x + \alpha y$, and therefore to Q,

$$\prod_r \left\{\frac{x + \alpha^{\pm h[1:r]} y}{Q}\right\} = \left\{\frac{\alpha^{\frac{\pm hkyq(k)}{x+y}}}{Q}\right\} \tag{16}$$

where $k = 2$, and the sign of h is selected so that $h[1:r] \not\equiv 1 \pmod{p}$. Now

$$\left\{\frac{\alpha}{Q}\right\} = \alpha^{\frac{N(Q)-1}{p}}$$

and $N(Q) = w$, where

$$\frac{x^p + y^p}{x + y} = w^p,$$

and by Furtwängler's* Theorem we have $w \equiv 1 \pmod{p^2}$, whence

$$\left\{\frac{\alpha}{Q}\right\} = 1$$

and from (16)

$$\prod_r \left\{\frac{x + \alpha^{\pm h[1:r]} y}{Q}\right\} = 1. \tag{16a}$$

Also, since

$$x + \alpha^i y = x + \alpha y + y\alpha(\alpha^{i-1} - 1),$$

the relation (16a) reduces to

$$\prod_r \left\{\frac{y(\alpha^{\pm h[1:r]-1} - 1)}{Q}\right\} = 1.$$

Equation (3) gives immediately

$$\left\{\frac{y(1-\alpha)}{Q}\right\} = 1.$$

* Wien. Ber. 1912, 121, Abt. IIa, p. 589.

Comparison of this with the previous relation gives

$$\prod_r \left\{\frac{\varepsilon}{Q}\right\} = 1 \tag{17}$$

where

$$\varepsilon = \frac{\alpha^{\pm h[1:r]-1}-1}{\alpha-1}.$$

Applying the relation (15) to (17), using

$$\frac{N(Q)-1}{p} \equiv 0 \pmod{p},$$

and expanding the result in powers of h, we have, if we note that terms not involving h all cancel,

$$\pm h A_1 R_1 + h^2 A_2 R_2 + \cdots + (\pm h)^{p-4} A_{p-4} R_{p-4} \equiv 0 \pmod{p}, \tag{18}$$

where

$$R_i \equiv \sum_{r=1}^{\mu} [1:r]^i$$

and the A's are expressions involving the B's and D's. The writer has proved*

$$\frac{(1-2^i)\,b_i}{2^{i-1}\,i} \equiv \sum_{r=1}^{\mu} r^{i-1} \pmod{p}, \tag{19}$$

where

$$b_1 \equiv -1/2 \quad b_{2a} = (-1)^{a+1} B_a, \qquad b_{2a+1} = 0, \qquad (a>0).$$

Hence $R_{2a} \equiv 0 \pmod{p}$, and (18) may be written

$$h A_1 R_1 + h^3 A_3 R_3 + \cdots + h^{p-4} A_{p-4} R_{p-4} \equiv 0 \pmod{p}. \tag{20}$$

Letting $h = 1, 2, \cdots, \mu-1$ in turn we obtain $\mu-1$ congruences. Considering them with respect to the quantities $A_1 R_1, A_3 R_3, \cdots, A_{p-4} R_{p-4}$, we have the determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2^3 & 2^5 & \cdots & 2^{p-4} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \mu-1, & (\mu-1)^3, & (\mu-1)^5, & \cdots, & (\mu-1)^{p-4} \end{vmatrix} = \prod_{i,j}^{1 \text{ to } \mu-1} (i+j)(i-j)\,((\mu-1)!)$$

* Ann. of Math., 2nd ser., vol. 18, p. 114, relation 11 for $k = 2$.

which is not divisible by p, and we obtain

$$A_j R_j \equiv 0 \pmod{p} \qquad (j = 1, 3, \cdots, p-4).$$

Using (19) we have

$$(1 - 2^{2n}) B_n A_{p-2n} \equiv 0 \pmod{p} \qquad (n = 2, 3, \cdots, \mu - 1).$$

Let $n = 2, 3, 4, 5$. These give, except for small values of p (for which values (1) is known to be impossible if xyz is prime to p),

$$B_n A_{p-2n} \equiv 0 \pmod{p} \qquad (n = 2, 3, 4, 5).$$

Writing down these A's in terms of the B's and D's we have

$$A_{p-4} \equiv \frac{(-1)^{a+1} B_a D_{3p}}{p} \equiv 0 \pmod{p}, \tag{21}$$

where $a = \dfrac{(p-4)p+1}{2}$. Similarly

$$A_{p-6} = \frac{(-1)^{a-p+1} B_{a-p} D_{5p}}{p} + \frac{(p-4)(p-5)}{2 \cdot 3} \cdot \frac{(-1)^{a+1} B_a D_{3p}}{p} \equiv 0 \pmod{p} \tag{21a}$$

and from (21) $B_a D_{3p} \equiv 0 \pmod{p^2}$ and therefore from (21a) $B_{a-p} D_{5p} \equiv 0 \pmod{p^2}$. Using $A_{p-8} \equiv A_{p-10} \equiv 0 \pmod{p}$ we have

$$B_{a-2p} D_{7p} \equiv B_{a-3p} D_{9p} \equiv 0 \pmod{p^2}. \tag{21b}$$

We shall now transform (21). We have

$$\frac{d \log (x + e^v y)}{dv} = \frac{e^v y}{x + e^v y}$$

and

$$\frac{e^v y}{x + e^v y} \cdot (x^{p^3} + e^{vp^3} y^{p^3}) = \sum_{i=1}^{p^3} (-1)^{i-1} e^{iv} y^i x^{p^3 - i}.$$

Differentiating the left hand member as a product and setting $v = 0$, we have

$$(x^{p^3} + y^{p^3}) D_{3p} \equiv \sum_{i=1}^{p^3} (-1)^{i-1} i^{3p-1} y^i x^{p^3 - i} \tag{22}$$

modulo p^2. Modulo p this reduces to

$$(x^{p^s}+y^{p^s})\,D_{sp} \equiv \left(\sum_{f=0}^{p(p-1)} x^{fp}y^{-fp}\right)\left(\sum_{i=1}^{p-1}(-1)^{i-1}\,i^{sp-1}\,y^i x^{p^s-i}\right) \pmod p$$

(23)

$$\equiv \frac{x^{p^s}+y^{p^s}}{x^p+y^p}\sum_{i=1}^{p-1}(-1)^{i-1}\,i^2\,y^i x^{p-i} \pmod p.$$

Also*

$$(-1)^{s\mu}\,\frac{B_{(sp+1)/2}}{(sp+1)/2} \equiv \frac{B_{(s+1)/2}}{(s+1)/2} \pmod p,$$

where $(s+1)/2$ is not a multiple of μ. This gives

$$(-1)^{\mu}\,\frac{B_a}{a} \equiv \frac{B_{\mu-1}}{\mu-1} \pmod p, \tag{24}$$

whence from (21) and (23), since $x+y \not\equiv 0 \pmod p$ and therefore $x^{p^s}+y^{p^s} \not\equiv 0 \pmod p$,

$$B_{\mu-1}f_s(x,y) \equiv 0 \pmod p, \tag{25}$$

where $f_r(x,y) = \sum_{i=1}^{p-1}(-1)^{i-1}\,i^{r-1}\,y^i x^{p-i}$. Similarly we find from (21a) and (21b)

$$B_{\mu-2}f_5(x,y) \equiv B_{\mu-3}f_7(x,y) \equiv B_{\mu-4}f_9(x,y) \equiv 0 \pmod p. \tag{25a}$$

Mirimanoff (l. c.) showed that we may take $f_i(x,y) \not\equiv 0 \pmod p$, $i = 3, 5, 7, 9$, whence $B_{\mu-1} \equiv B_{\mu-2} \equiv B_{\mu-3} \equiv B_{\mu-4} \equiv 0 \pmod p$. Using $B_{\mu-1} \equiv 0 \pmod p$ we have from (24)

$$B_a \equiv 0 \pmod p.$$

Write (21) in the form

$$B_a\,D_{sp} \equiv 0 \pmod{p^2}.$$

Since $B_a \equiv 0 \pmod p$, D_{sp} may be reduced modulo p and (23) gives

$$B_a\,f_s(x,y) \equiv 0 \pmod{p^2}$$

and therefore $B_a \equiv 0 \pmod{p^2}$ since $f_s(x,y) \not\equiv 0 \pmod p$, as we have already seen. Similarly

$$B_a \equiv B_{a-p} \equiv B_{a-2p} \equiv B_{a-3p} \equiv 0 \pmod{p^2},$$

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* Vandiver, Bull. Amer. Math. Soc., vol. 25 (1919), p. 461. There is a misprint in the second congruence involving the b's on p. 461; in lieu of $s(l+1)/2$ read $(sl+1)/2$.

whence the theorem:

If $x^p+y^p+z^p=0$ *is satisfied in integers* x, y, z *prime to the odd prime,* p, *then*

$$B_s \equiv 0 \pmod{p^2}$$

$s=(lp+1)/2$, $l=p-4, p-6, p-8, p-10$.

This result was previously given by the writer* without proof. Using (14b) for $k=2$ in connection with the theorem we have

$$\frac{(-1)^{l/2}B_{l/2}(2^l-1)}{2^{l-1}l}=\sum_{r=1}^{\mu} r^{l-1} \equiv 0 \pmod{p^2}$$

$=(2m-1)p+1$, $m=\mu-1, \mu-2, \mu-3, \mu-4$. These relations may also be written

$$\sum_{r=1}^{\mu}\frac{1}{r^{ap}} \equiv 0 \pmod{p^2} \qquad (a=3, 5, 7, 9).$$

5. Putting $i=p^s(2m-1)+1$ in (12a) we have

$$\frac{(n^i-1)\sum_{a=1}^{p-1}a^i}{pi} \equiv \sum_{a=1}^{p-1} y_a a^{i+1} \pmod{p^2}. \tag{27}$$

We have also†

$$\sum_{a=1}^{p-1} n^i \equiv (-1)^{i/2-1} p B_{i/2} \pmod{p^3}.$$

Comparing with (27) we have

$$\frac{(n^i-1)(-1)^{i/2-1}B_{i/2}}{i}=\sum_{a=1}^{p-1} y_a a^{i-1} \pmod{p^2}.$$

From (14a) we also have

$$\frac{(-1)^{l/2-1}B_{l/2}(n^l-1)}{l}=\sum_{a=1}^{p-1} y_a a^{l-1} \pmod{p^2}.$$

Since $a^{i-1} \equiv a^{l-1} \pmod{p^2}$ for $a=1, 2, \cdots, p-1$, the last two congruences give

$$\frac{(-1)^{i/2-1}(n^i-1)B_{i/2}}{i} \equiv \frac{(-1)^{l/2-1}B_{l/2}(n^l-1)}{l} \pmod{p^2}.$$

* Bull. Amer. Math. Soc., vol. 24 (1918), p. 472; vol. 28 (1922), p. 259.

† Vandiver, Bull. Amer. Soc., vol. 25, p. 460, formula (5) for $s=2m-1$, $a=8$, $l=p$.

For $m = \mu - 1$, $\mu - 2$, $\mu - 3$, $\mu - 4$, we have from (26), since $n^i - 1 \not\equiv 0 \pmod{p}$ if n is a primitive root of p

$$B_{i/2} \equiv 0 \pmod{p^2}, \tag{28}$$

for $i = tp^s + 1$, $t = p-4$, $p-6$, $p-8$, $p-10$.

The first factor h of the class number* has the property

$$h \equiv \frac{\prod p(-1)^{(sl^s-1)/2} B_{(sp^s+1)/2}}{2^{\frac{1}{2}(p-3)}} \pmod{p^s},$$

$s = 1, 3, \cdots, p-2$. Applying relation (28) to this we have

$$h \equiv 0 \pmod{p^s}.$$

Whence the theorem:

If $x^p + y^p + z^p = 0$ is satisfied in integers prime to the odd prime p, then the first factor of the class number of the field $\Omega(e^{2i\pi/p})$ is divisible by p^s.

Kummer† computed the first factor h of the class number of the field $\Omega(e^{2i\pi/p})$ for all values of p less than 167 and found that none were divisible by p except $p = 37$, 59, 67, 101, 103, 131, 149, and 157. All these were divisible by p but not by p^2 excepting for 157, when this first factor is divisible by p^2 but not by p^3. Also it follows from these computations and my expression for h, modulo p^n, already referred to, that

$$B_s \not\equiv 0 \pmod{p^2},$$

$s = (lp+1)/2$, $l = p-4, p-6, p-8, p-10$, for any $p < 167$.

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* Vandiver, Bull. Am. Math. Soc., vol. 25 (1918), p. 460, formula (8) for $a = 8$, $l = p$.

† Berlin. Monatsberichte, 1874, pp. 239–248.

CORNELL UNIVERSITY

Sept. 1923.

ON AN INFINITE SYSTEM OF NON-ABELIAN GROUPS OF ORDER nm^n.*

By W. E. Edington.

Consider two operators s_1 and s_2 having the property that the transforms of their product $s_1 s_2$ by s_1 and its powers are all distinct and commutative. Let the order of s_1 be n. Then the number of distinct transforms of $s_1 s_2$ by s_1 and its powers is n.†

Let

$$s_2^{-1} s_1 s_2 = t \qquad \text{and} \qquad s_1^{-1} s_2 s_1 = p.$$

Then

$$(1) \qquad s_2^{-x} s_1 s_2^{x} = t^x s_1^{-(x-1)} \quad \text{and} \quad s_1^{-\tau} s_2 s_1^{\tau} = p^{\tau} s_2^{-(\tau-1)},$$

for, if the first of these two relations be assumed to be true, we have

$$s_2^{-(x+1)} s_2 s_1^{x+1} = s_2^{-1} \cdot s_2^{x} s_1 s_2^{x} \cdot s_2 = s_2^{-1} t^x s_1^{-(x-1)} s_2 = s_2^{-1} \cdot s_2^{-1} s_1^{x} s_2 \cdot s_1^{-(x-1)} s_2$$

$$= s_2^{-2} \cdot s_1^{x} s_2 s_1 s_1^{-x} \cdot s_2 s_1 \cdot s_1^{-1} = s_2^{-2} \cdot s_2 s_1 \cdot s_1^{x} s_2 s_1 s_1^{-x} \cdot s_1^{-1} = s_2^{-1} s_1^{x+1} s_2 s_1^{-x}$$

$$= t^{x+1} s_1^{-x}.$$

It is easy to show that the above relation holds for $x = 1, 2$, and hence it holds for all values of x. The second relation in (1) is proved in a similar manner.

Since $t^n = 1$, it follows for $x = n$ that s_2^n and s_1 are commutative, and hence s_2^n is commutative with $s_1 s_2^{\beta}$, and s_2^n is the lowest power of s_2 that is commutative with s_1, unless it is assumed that some power of t is equal to the same power of s_1. In order to get the system of groups of highest order it will be assumed that no power of t equals a power of s_1 except the identity. Hence it follows that *the order of s_2 must be divisible by n,* for, if not, s_2^n would generate s_2 and then s_1 and s_2 would be commutative contrary to the hypothesis that s_1 and s_2 are not commutative.

The transform of $s_1 s_2$ by s_2 and its powers are all commutative, for

$$s_2^{-\alpha} s_1 s_2 s_2^{\alpha} \cdot s_2^{-\varphi} s_1 s_2 s_2^{\varphi} = s_2^{-\alpha+1} \cdot s_2^{-1} s_1 s_2 \cdot s_2^{\alpha-\varphi} s_1 s_2^{\varphi-\alpha} \cdot s_2^{\alpha+1}$$

$$= s_2^{-\alpha+1} t^{\varphi-\alpha+1} s_1^{\alpha-\varphi} \cdot s_1 s_2^{\alpha+1} = s_2^{-\alpha+1} \cdot s_2^{-(\varphi-\alpha+1)} s_1 s_2^{\varphi-\alpha+1} \cdot s_1 s_2^{\alpha+1}$$

$$= s_2^{-\varphi} s_1 s_2 s_2^{\varphi} \cdot s_2^{-\alpha} s_1 s_2 s_2^{\alpha}.$$

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* Presented to the American Mathematical Society, December 29, 1922.

† Miller, Blichfeldt and Dickson, Finite Groups, pp. 58-59.

If, in the second relation of (1), one sets $\tau = n$, it follows that $p^n = s_2^n$, and hence p^n and s_1 are commutative.

Since the order of s_2 is divisible by n, let the order of s_2 be mn. Then, since s_2^n is commutative with s_1, all transforms of $s_1 s_2$ by powers of s_2 equal to or greater than n are reducible to transforms of $s_1 s_2$ by powers of s_2 lower than n. Now

$$s_1^{-\delta} s_1 s_2 s_1^{\delta} = s_1 s_1^{-\delta} s_2 s_1^{\delta} = s_1 p^{\delta} s_2^{1-\delta} = s_1 \cdot s_1^{-1} s_2^{\delta} s_1 \cdot s_2^{1-\delta} = s_2^{\delta} s_1 s_2 s_2^{-\delta},$$

that is, the transforms of $s_1 s_2$ by s_2 and its powers are equal to the transforms of $s_1 s_2$ by s_1 and its powers in some arrangement, or the two sets of transforms are identical. Therefore the transforms of $s_1 s_2$ by s_2 and its powers are commutative with the transforms of $s_1 s_2$ by s_1 and its powers.

The preceding results are summarized in the following theorem.

THEOREM I. *If s_1 and s_2 are two operators having the property that the transforms of their product $s_1 s_2$ by s_1 and its powers are all distinct and commutative, then $s_2^{-x} s_1 s_2^{x} = t^x s_1^{-(x-1)}$ and $s_1^{-\tau} s_2 s_1^{\tau} = p^{\tau} s_2^{-(\tau-1)}$, where $s_2^{-1} s_1 s_2 = t$ and $s_1^{-1} s_2 s_1 = p$, and the set of transforms of $s_1 s_2$ by s_1 and its powers is identical with the set of transforms of $s_1 s_2$ by s_1 and its powers.*

Consider next the following general relations:

$$(2) \qquad \begin{aligned} s_1^{-1} s_1 s_2 s_1 \cdot s_1^{-2} s_1 s_2 s_1^2 \cdot s_1^{-3} s_1 s_2 s_1^3 \cdots s_1^{-\varphi} s_1 s_2 s_1^{\varphi} &= s_2^{\varphi} s_1^{\varphi}, \\ s_1^{\varepsilon} s_2 s_1 s_1^{-\varepsilon} \cdot s_1^{\varepsilon-1} s_2 s_1 s_1^{1-\varepsilon} \cdot s_1^{\varepsilon-2} s_2 s_1 s_1^{2-\varepsilon} \cdots s_1 s_2 s_1 s_1^{-1} &= s_1^{\varepsilon} s_2^{\varepsilon}. \end{aligned}$$

The left members of these relations are composed of products of transforms of $s_1 s_2$ by s_1 and its powers and hence are commutative, and therefore $s_2^{\varphi} s_1^{\varphi}$ and $s_1^{\varepsilon} s_2^{\varepsilon}$ are commutative for all values of φ and ε. Furthermore, since these transforms are all commutative and of the same order as $s_1 s_2$, the orders of $s_1^{\varepsilon} s_2^{\varepsilon}$ aud $s_2^{\varphi} s_1^{\varphi}$ are equal to the order of $s_1 s_2$.

The order of s_1 is n and the order of s_2 is mn. Let $\alpha + \beta = n$. Then

$$\begin{aligned} s_1^{\alpha} (s_1^{\beta} s_2^{\beta})^m \cdot (s_2^{\alpha} s_1^{\alpha})^m s_1^{-\alpha} &= s_1^{\alpha} \cdot s_1^{\beta} s_2^{\beta} \cdot s_2^{\alpha} s_1^{\alpha} \cdot s_1^{\beta} s_2^{\beta} \cdot s_2^{\alpha} s_1^{\alpha} \cdots s_2^{\alpha} s_1^{\alpha} \cdot s_1^{-\alpha} \\ &= (s_1^{\alpha+\beta} s_2^{\alpha+\beta})^m = (s_1^n s_2^n)^m = s_2^{mn} = 1. \end{aligned}$$

Therefore $(s_1^{\beta} s_2^{\beta})^m \cdot (s_2^{\alpha} s_1^{\alpha})^m = 1$. If m is greater than unity $s_2^{\alpha} s_1^{\alpha}$ can not be the inverse of $s_1^{\beta} s_2^{\beta}$, for, since the order of s_2 is mn,

$$s_2^{\alpha} s_1^{\alpha} \neq s_2^{\alpha-n} s_1^{\alpha-n} = s_2^{\alpha-n} s_1^{\alpha} = s_2^{-\beta} s_1^{-\beta}.$$

Since $s_1^\beta s_2^\beta$ and $s_2^\alpha s_1^\alpha$ are commutative and are not inverses for m greater than unity it follows that their common order is m or a divisor of m. Since it is desired to find the group of greatest order generated by s_1 and s_2, the order of $s_1^\beta s_2^\beta$, and hence also of $s_1 s_2$, will be taken equal to m.

It is to be observed that no condition has been imposed restricting the commutativity of the operators $s_1^\varepsilon s_2^\varepsilon$ and $s_2^\varphi s_1^\varphi$. Thus any operator of the form $s_1^\nu s_2^\mu$, $\mu \equiv \nu$, mod. n, belongs to the general set of commutative operators.

From the preceding results follows

THEOREM II. *If s_1 and s_2 are two operators having the property that the transforms of $s_1 s_2$ by s_1 and its powers are all distinct and commutative, and if $s_1^n = s_2^{mn} = 1$, then the greatest order that $s_1 s_2$ may have is m, and all products of the forms $s_1^\varepsilon s_2^\varepsilon$, $s_2^\varphi s_1^\varphi$, and $s_1^\mu s_2^\nu$, $\nu \equiv \mu$, mod. n, are commutative.*

The operators $p^\varepsilon t^\varepsilon$ and $p^\varphi t^\varphi$ are commutative, for, since $p^\varepsilon = s_1^{-1} s_2^\varepsilon s_1$. and $t^\varepsilon = s_2^{-1} s_1^\varepsilon s_2$, etc., we have the product $p^\varepsilon t^\varepsilon p^\varphi t^\varphi$ expressible as a product of the commutative operators $s_1 s_2$, $s_1^{-1} s_2^{-1}$, $s_1^{-\varepsilon} s_2^{-\varepsilon}$, $s_1^{-\varphi} s_2^{-\varphi}$, $s_2^{\varepsilon-1} s_1^{\varepsilon-1}$, $s_2^{\varphi-1} s_1^{\varphi-1}$, $s_2^{\varepsilon+1} s_1^{\varepsilon+1}$, $s_2^{\varphi+1} s_1^{\varphi+1}$, the first two of these operators each being used twice, and on account of their commutativity they may be arranged so that their product also equals $p^\varphi t^\varphi p^\varepsilon t^\varepsilon$. Moreover, since $p^\varepsilon t^\varepsilon$ is reducible to the product of five commutative operators all of order m, the order of $p^\varepsilon t^\varepsilon$ is also m. Furthermore, p is of order m and t is of order n, and hence p and t will generate the same group as s_1 and s_2, provided the transforms of pt by t and its powers are distinct and commutative.

Since

$$s_1^{-\varepsilon} s_1 s_2 s_1^\varepsilon \cdot s_1^{-\varphi} s_1 s_2 s_1^\varphi = s_1^{-\varphi} s_1 s_2 s_1^\varphi \cdot s_1^{-\varepsilon} s_1 s_2 s_1^\varepsilon,$$

then

$$(3) \qquad \begin{aligned} s_1^{1-\varepsilon} s_2 s_1^{\varepsilon-\varphi+1} s_2 s_1^\varphi &= s_1^{1-\varphi} s_2 s_1^{\varphi-\varepsilon+1} s_2 s_1^\varepsilon, \\ s_1^{\varphi-\varepsilon} s_2 s_1^{\varepsilon-\varphi} \cdot s_1 s_2 &= s_2 s_1 \cdot s_1^{\varphi-\varepsilon} s_2 s_1^{\varepsilon-\varphi}. \end{aligned}$$

Now $s_1^{\varphi-\varepsilon} s_2 s_1^{\varepsilon-\varphi}$ is a transform of s_2 by a power of s_1 and, since φ and ε take all values from 1 to $n-1$ independently, this represents any desired transform of s_2 by a power of s_1. Hence the relation (3) shows that, if the group generated by s_1 and s_2 is to be non-abelian, no transform of s_2 by s_1 and its powers may be commutative with $s_1 s_2$, and in a similar manner it can be shown that neither $s_1 s_2$ nor $s_2 s_1$ may be commutative with any transform of s_1 by s_2 and its powers or of s_2 by s_1 and its powers.

The order of the group generated by s_1 and s_2 under the conditions found above will now be determined and the existence of a group for every value of m and n will be proved.

Since s_1 transforms $s_1 s_2$ into n distinct operators which are commutative, the group of greatest order which these transforms may generate is an abelian group of order m^n, which will be designated by H. H is invariant under s_1 since it is generated by $s_1 s_2$ and the transforms of $s_1 s_2$ by s_1 and its powers, and H is an invariant under s_2 since

$$s_2^{-1} \cdot s_1^{-\varepsilon} s_2 s_1^{\varepsilon} \cdot s_2 = s_2^{-1} s_1^{-1} \cdot s_1^{1-\varepsilon} s_1 s_2 s_1^{\varepsilon-1} \cdot s_1 s_2,$$

which is in H. Hence H is an invariant subgroup of the group G generated by s_1 and s_2. No power of s_1, except the identity, is in H, for if s_1^{α} were in H, then s_1^{α} and $s_1 s_2$ would be commutative and s_1 would not then transform $s_1 s_2$ into n distinct operators. Therefore s_1 and H generate a group G of order at most nm^n. s_2 is in this group since $s_1^{-1} \cdot s_1 s_2 = s_2$.

To prove the existence of the group, let

$$s_1 = abc \cdots def \cdot ghi \cdots jkl \cdots\cdots mno \cdots pqr \cdot stu \cdots vwx,$$

composed of m cycles each of order n, and let

$$s_2 = axwv \cdots utsrqp \cdots onm \cdots lkj \cdots ihgfed \cdots cb,$$

which is of order mn. Then

$$s_1 s_2 = fxr \cdots 1,$$

which is of order m and is seen to be formed by taking the last letter of each cycle of s_1, but in the reserved cyclic order in which they occur in s_1. Also

$$\begin{aligned} s_1^{-1} s_1 s_2 s_1 &= asm \cdots g, \\ s_1^{-2} s_1 s_2 s_1^2 &= btn \cdots h, \\ &\cdots\cdots\cdots, \\ s_1^{-(n-1)} s_1 s_2 s_1^{(n-1)} &= ewq \cdots k. \end{aligned}$$

It is evident that these $n - 1$ transforms of $s_1 s_2$ by s_1 and its powers fulfil the general conditions and hence they generate an abelian group of order m^n. By adding another cycle of order n to s_1 and changing s_2 to correspond, it is evident by mathematical induction that these two substitutions fulfil the general conditions and generate a group of order nm^n. When $n = 2$, $m = 2$, G is the octic group. For $n = 2$, $m = 3$, G is the group of order 18,

otherwise defined by the conditions $s_1^2 = s_2^6 = (s_1 s_2)^3 = (s_1 s_2^3)^3 = 1$, $s_1 s_2^2 = s_2^2 s_1$, having six operators of order 6 and eight operators of order 3. When $n = 3$, $m = 2$, the group defined is of order 24, also defined by the conditions $s_1^3 = s_2^6 = (s_1 s_2)^2 = (s_1^2 s_2^2)^2 = 1$, and has eight operators of order 6, eight operators of order 3, and seven operators of order 2. Since the generators are known as substitutions for every integral value of m and n, it follows that the existence of this infinite system of groups has been established.

THEOREM III. *If s_1 and s_2 are two operators having the property that the transforms of $s_1 s_2$ by s_1 and its powers are all distinct and commutative, and if they fulfil the conditions*

$$s_1^n = s_2^{mn} = (s_1 s_2)^m = 1,$$

they generate a group of order $n m^n$, and a group exists for every value of m and n.

The substitutions representing s_1 and s_2 possess an interesting property. In the substitutions representing $s_1^\alpha s_2^\alpha$, $\alpha = 1, 2, \cdots, (mn-1)$, it will be found that $s_1 s_2$ and $s_2 s_1$ are each composed of one cycle, $s_1^2 s_2^2$ and $s_2^2 s_1^2$ each of two cycles, and so on, and $s_1^n s_2^n$ and $s_2^n s_1^n$ each of n cycles, all cycles being of order m. The operators of the form $s_1^\mu s_2^\nu$, $\nu \equiv \mu$, mod. n, $n \leqq \nu \leqq (m-1)n$, are composed of n cycles each, but some or all of these cycles will be powers of the cycles in $s_1^n s_2^n$. All the operators of the form $s_1^\mu s_2^\nu$, $\nu \equiv \mu$, mod. n, $(m-1)n < \nu \leqq mn$, will be represented by substitutions composed of less than n cycles, the inverses of the first set.

The two sets of operators $s_1^\alpha s_2^\alpha$ and $p^\varepsilon t^\varepsilon$, $\alpha, \varepsilon = 1, 2, \cdots, mn$, will now be considered. The set $p^\varepsilon t^\varepsilon$ is composed of commutative operators. For $\varepsilon > n$ the operators take the form $p^\varepsilon t^\eta$, $\eta \equiv \varepsilon$, mod. n. It is to be observed that p^n is one of the set and it has already been proved that $s_2^n = p^n$ is commutative with s_1, and therefore it is commutative with $t = s_2^{-1} s_1 s_2$. This property also follows independently out of the commutativity of the set, for upon setting $\varepsilon = n$ in the relation $p^\varepsilon t^\varepsilon p^\varphi t^\varphi = p^\varphi t^\varphi p^\varepsilon t^\varepsilon$, it follows that $p^{\varphi+n} t^\varphi = p^\varphi t^\varphi p^n$ or $p^n t^\varphi = t^\varphi p^n$, or p^n is commutative with t, since $\varphi = 1, 2, \cdots, mn$. From this last property it follows that all operators of the form $p^\varepsilon t^\eta$, $\eta \equiv \varepsilon$, mod. n, are generated by operators of the form $p^\eta t^\eta$, $\eta = 1, 2, \cdots, (n-1)$ and p^n, that is, the set $p^\varepsilon t^\varepsilon$, $\varepsilon = 1, 2, \cdots, mn$, will generate the same group as p^n and the set $p^\eta t^\eta$, $\eta = 1, 2, \cdots, (n-1)$, together generate. The group generated by the set $p^\varepsilon t^\varepsilon$ has then at most n distinct independent generators of the form $p^\varepsilon t^\varepsilon$, that is, a number equal to the order of t. The author has proved in another paper* that if two operators t_1 and t_2 fulfil the conditions

* Annals of Math., vol. 25 (1923), pp. 85–90.

$$t_1^n = t_1^n = (t_1^\alpha t_2^\alpha)^m = 1, \quad t_1^\alpha t_2^\alpha t_1^\beta t_2^\beta = t_1^\beta t_2^\beta t_1^\alpha t_2^\alpha, \quad (\alpha, \beta = 1, 2, \cdots, (n-1)),$$

the group of highest order generated by t_1 and t_2 is of order nm^{n-1}, and this group exists for all values of m and n. It is evident that this group is the same as the one generated by s_1 and s_2^m or by p^m and t. Properties of this group and its operators are given in the other paper. However, these relations may be summarized in the following theorem which is merely a more generalized theorem. The proof is exactly like the proof in the paper already referred to.

THEOREM IV. *If two operators s_1 and s_2 fulfil the conditions*

$$s_1^n = s_2^{mn} = (s_1^\alpha s_2^\alpha)^m = 1, \quad s_1^\alpha s_2^\alpha s_1^\beta s_2^\beta = s_1^\beta s_2^\beta s_1^\alpha s_2^\alpha, \quad (\alpha, \beta = 1, 2, \cdots, mn),$$

the group of greatest order generated by s_1 and s_2 is of order nm^n, and such a group exists for every value of m and n.

PURDUE UNIVERSITY.

CONFORMAL AND GEODESIC MAPPING.

By Arthur Bramley.

1. We shall first consider the problem of finding the relations which exist between the coefficients of the linear elements of two varieties immersed in different enveloping spaces, and, from these, the relations among the Christoffel symbols and curvature tensors, the varieties being defined by the same set of functions in terms of the coordinates of the respective enveloping spaces.

Consider a V_n immersed in an enveloping V_{n+k}, then, if V_n is defined by the set of equations

$$x^l = \varphi^l(u_1, \cdots, u_n) \qquad (l = 1, \cdots, n+k) \tag{1}$$

the linear element of V_n is

$$ds^2 = a_{rs}\,\varphi^r_\alpha\,\varphi^s_\beta\,du^\alpha\,du^\beta, \tag{2}$$

where the linear element of V_{n+k} is

$$ds^2 = a_{rs}\,dx^r\,dx^s \tag{3}$$

and a subscript, as in φ^r_α, denotes differentiation with respect to u^α.

Now make a transformation of the coordinates in V_{n+k}, letting

$$x^l = f^l(y_1, \cdots, y_{n+k}); \tag{4}$$

then the linear element of the transformed space V'_{n+k} is

$$ds^2 = a_{rs} f^r_l f^s_m\,dy^l\,dy^m, \tag{5}$$

and the linear element of the variety V'_n in V'_{n+k} defined by the set of equations

$$y^l = \varphi^l(u_1, \cdots, u_n),$$

where the functions φ^l are of the same form as in (1), is

$$ds^2 = a_{rs} f^r_l f^s_m \varphi^l_\alpha \varphi^m_\beta\,du^\alpha\,du^\beta = a_{\alpha\beta}\,du^\alpha\,du^\beta, \tag{6}$$

where

$$a_{\alpha\beta} = f^r_l f^s_m \varphi^l_\alpha \varphi^m_\beta\,a_{rs}. \tag{7}$$

The Christoffel symbols of the first kind formed with respect to (7) are

$$(9)\qquad \Gamma_{\alpha\beta,\delta} = \left[\bar{\Gamma}_{rs,k} f_l^r f_m^s \varphi_\alpha^l \varphi_\beta^m + a_{rs} f_k^s f_{lm}^r \varphi_\alpha^l \varphi_\beta^m + a_{rs} f_k^r f_m^s \varphi_{\alpha\beta}^m\right] \varphi_\delta^k,$$

where

$$\bar{\Gamma}_{rs,k} = \frac{1}{2}\left(\frac{\partial a_{rk}}{\partial \varphi_s} + \frac{\partial a_{sk}}{\partial \varphi_r} - \frac{\partial a_{rs}}{\partial \varphi_k}\right)$$

is Christoffel's symbol for the space whose linear element is

$$(11)\qquad ds^2 = a_{rs}(\varphi)\, d\varphi^r\, d\varphi^s,$$

or, more concisely, the Christoffel symbols formed for the linear element (3) with the x's replaced by their values in terms of the u's by (1).

The equations of the geodesics of V_n', namely,

$$(12)\qquad a_{\beta\delta} \frac{d^2 u^\beta}{ds^2} + \Gamma_{\alpha\beta,\delta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = 0,$$

become

$$(13)\qquad \begin{aligned} a_{rs} f_l^r f_m^s \varphi_\beta^l \varphi_\delta^m \frac{d^2 u^\beta}{ds^2} + \varphi_s^t \{ & \bar{\Gamma}_{rs,t} \varphi_\alpha^l \varphi_\beta^m f_l^r f_m^s + a_{rs} f_t^s \varphi_\alpha^l \varphi_\beta^m f_{lm}^r \\ & + a_{rs} f_t^r f_m^s \varphi_{\alpha\beta}^m \} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = 0 \end{aligned}$$

in terms of the enveloping space. If the enveloping space is Euclidean, then $\bar{\Gamma}_{rs,t} = 0$ and the equations of the geodesics become

$$(14)\qquad \sum_r f_l^r f_m^r \varphi_\beta^l \varphi_\delta^m \frac{d^2 u^\beta}{ds^2} + \varphi_s^t \left\{ \sum_r f_l^r f_{lm}^r \varphi_\beta^m \varphi_\alpha^l + f_t^r f_m^r \varphi_{\alpha\beta}^m \right\} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = 0.$$

If we further restrict the problem by taking the variety V_n as Euclidean by choosing

$$(15)\qquad \varphi_\alpha^l = \delta_\alpha^l, \qquad \delta_\alpha^l = \begin{cases} 0, & \alpha \neq l \\ 1, & \alpha = l \end{cases} \quad (l = 1, \cdots, n+k;\ \alpha = 1, \cdots, n)$$

that is, $x^1 = u^1, \cdots, x^n = u^n$, while $x^{n+l} = 0\ (l = 1, \cdots, k)$, then the equations of the geodesics in the variety V_n' become

$$(16)\qquad \sum_r f_\alpha^r f_\beta^r \frac{d^2 u^\alpha}{ds^2} + \left\{ \sum_r f_\beta^r f_{\alpha\delta}^r \right\} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} = 0.$$

If we are given a set of functions $\alpha_{\alpha\beta}$ defining the linear element (7), we can consider this space as a variety V_n' immersed in an enveloping V_{n+k}' through the functions f and φ defined by (7), the order of the enveloping space satisfying the condition $k \leqq n(n-s)/4$.

If we consider the two varieties immersed in V_{n+k}, namely $x^l = \varphi^l(u_1, \cdots, u_n)$ and $y^l = \varphi^l(u_1, \cdots, u_n)$, then the above transformation gives a means of comparison between the spaces whose linear elements are

$$ds^2 = a_{rs}\varphi^r_\alpha \varphi^s_\beta \, du^\alpha \, du^\beta = \beta_{\alpha\beta} \, du^\alpha \, du^\beta, \tag{17}$$

and

$$ds^2 = a_{rs} f^r_l f^s_m \varphi^l_\alpha \varphi^m_\beta \, du^\alpha \, du^\beta = \alpha_{\alpha\beta} \, du^\alpha \, du^\beta. \tag{7}$$

Since the two varieties whose fundamental forms are $\alpha_{\alpha\beta}$ and $\beta_{\alpha\beta}$, respectively, can be considered as immersed in the same space V_{n+k} with the form a_{rs}, the one being transformed into the other by a point transformation of the enveloping space, the relationship between them is reciprocal.

If we eliminate the φ's in the f's, expressing the f's as functions of the u's, then

$$\alpha_{\alpha\beta} = a_{rs} \bar{f}^r_\alpha \bar{f}^s_\beta, \qquad \beta_{\alpha\beta} = a_{rs} \varphi^r_\alpha \varphi^s_\beta,$$

where $\bar{f}^r_\alpha = f^r_\beta \varphi^\beta_\alpha$, so that the equations deduced will express the converse relationship by replacing f's and φ's, the original variety being defined by the equation $x^l = f^l$, while the point transformation becomes $x^l = \varphi^l(y_1, \cdots, y_{n+k})$.

We shall now consider the question of geodesic mapping and conformal representation in the case where one of the varieties is Euclidean as is also the enveloping space.

2. **Conformal representation of a V_n on an E_n in an E_{n+k}.** Since $a_{rs} = \delta_{rs}$ and $\beta_{\alpha\beta} = \delta_{\alpha\beta}$, we have

$$\alpha_{\alpha\beta} = f^r_l f^l_m \varphi^l_\alpha \varphi^m_\beta, \qquad \delta_{\alpha\beta} = \varphi^r_\alpha \varphi^r_\beta.$$

If we set

$$\varphi^l_\alpha = \delta^l_\alpha / \sqrt{n}, \qquad \varphi^{n+i}_\alpha = 0, \qquad (l = 1, \cdots, n;\ i = 1, \cdots, k)$$

then

$$\alpha_{\alpha\beta} = \frac{1}{n} f^r_\alpha f^r_\beta, \tag{18}$$

$$\Gamma_{\alpha\beta,\delta} = \frac{1}{n} f^r_\delta f^r_{\alpha\beta}. \tag{19}$$

Since $a_{\alpha\beta}$ defined by (18) can be regarded as the coefficient in the linear element of any n-space, we see that (the product of) Maschke's symbolic function* defined by

$$a_{ij} = \bar{f}_i \bar{f}_j,$$

where

$$\Gamma_{kl,i} = \bar{f}_i \bar{f}_{kl},$$

can be expressed as the sum of the product of $n+k$ real functions, $n+k$ being the order of the enveloping space, namely,

$$n \bar{f}_i \bar{f}_j = \sum_{r=1}^{n+k} f_i^r f_j^r. \tag{20}$$

In the case where $f^1 = f^2 = \cdots = f^{n+k} = f$ then Maschke's symbolic functions are real and equal to f, that is, all the coordinates undergo the same transformation.

We shall next express the curvature of the space V_n' with the linear element (18) in terms of the functions f^i.

$$R_{ik} = \frac{\partial}{\partial x^r} \Gamma^r_{ik} - \frac{\partial}{\partial x^k} \Gamma^r_{ir} + \Gamma^r_{ik} \Gamma^s_{rs} - \Gamma^s_{ir} \Gamma^r_{ks},$$

and the scalar curvature is given by $R = g^{ik} R_{ik}$, where $g_{ik} = f_i^a f_k^a / n$ and $g_{ik} g^{il} = \delta^l_k$. Simplifying the above formulae for the case first considered, we obtain

$$n R_{ik} = g^{lr} \{ f^a_{lr} f^a_{ik} - f^a_{lk} f^a_{ir} + f^a_i f^a_{ikr} - f^a_l f^a_{irk} \}$$
$$+ \frac{g^{lr} g^{ms}}{n} \{ f^b_{ik} f^b_l f^a_{rs} f^a_m - f^b_{ir} f^b_m f^a_{ks} f^a_l \} + f^a_{ik} f^a_l \frac{\partial g^{lr}}{\partial x^r} - f^a_{ir} f^a_l \frac{\partial g^{lr}}{\partial x^k}.$$

If the two spaces are to be in conformal correspondence, then

$$ds^2 = a_{\alpha\beta} \, du^\alpha \, du^\beta$$

must be of the form

$$ds^2 = \varrho(u_1, \cdots, u_n)(du_1^2 + du_2^2 + \cdots + du_n^2), \tag{a}$$

so that

$$n\varrho = f^r_\alpha f^r_\alpha, \qquad 0 = f^r_\alpha f^r_\beta \quad (\alpha \neq \beta).$$

* Trans. Amer. Math. Soc., vol. 1 (1900), p. 197, and vol. 4 (1903), p. 445.

Since there are $n(n+1)/2$ equations defining $n+k$ functions, the above equations always have a solution for $f^1, \cdots, f^{n+k}$, where $n+k$ is the order of the enveloping space.

Now from the form (a) we obtain

$$\Gamma_{\alpha\beta\delta} = 0, \qquad \Gamma_{\alpha\beta\beta} = n\frac{\partial \varrho}{\partial x^\alpha}.$$

$$\Gamma_{\alpha\alpha\beta} = -n\frac{\partial \varrho}{\partial x^\beta}, \qquad \Gamma_{\alpha\alpha\alpha} = n\frac{\partial \varrho}{\partial x^\alpha},$$

$$\alpha^{\alpha\beta} = \frac{\delta_{\alpha\beta}}{\alpha_{\alpha\beta}} = \frac{\delta_{\alpha\beta}}{n\varrho},$$

so that

$$R_{\alpha\beta} = -n\frac{\partial^2 \log \varrho}{\partial u^\alpha \partial u^\beta}$$

and the curvature becomes

$$\text{(b)} \quad R = \alpha^{\alpha\beta} R_{\alpha\beta} = -\frac{1}{\varrho}\left\{\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \cdots + \frac{\partial^2}{\partial u_n^2}\right\}\log \varrho = -\frac{1}{\varrho}\nabla^2 \log \varrho.$$

This expression gives the curvature of the space that can be mapped conformally on a Euclidean space.

If we wish to find the type of transformation which characterizes conformal representation, we must inquire about the character of the solutions of the equations

$$\varrho = [f_\alpha^r]^2, \qquad 0 = f_\alpha^r f_\beta^r.$$

We shall only consider the case where ϱ is a polynomial, although other cases can be treated in an analogous manner. Let

$$\begin{aligned} x^i &= (A_{i1}u_1 + A_{i2}u_2 + \cdots + A_{in}u_n)^p = A_i^p \quad (i = 1, \cdots, n) \\ x^{n+j} &= (B_{j11}u_1 + B_{j12}u_2 + \cdots + B_{j1n}u_n)A_1^{p-1} \\ &\quad + (B_{j21}u_1 + \cdots + B_{j2n}u_n)A_2^{p-1} + \cdots \\ &\quad + (B_{jn1}u_1 + \cdots + B_{jnn}u_n)A_n^{p-1} \\ &= (B_{ji})A_i^{p-1} \quad (j = 1, \cdots, l). \end{aligned}$$

Taking the case where $p = 1/2$, we find that the equations

$$\bar{A}_1 x_1^2 + \bar{A}_2 x_2^2 + \cdots + \bar{A}_n x_n^2 = 1,$$

$$x^{n+j} = \sum_i B_{j1i} x_i^2/x_1 + \sum_i B_{j2i} x_i^2/x_2 + \cdots + \sum_i B_{jni} x_i^2/x_n, \quad (j = 1, \cdots, l),$$

determine the nature of the second space as viewed from the enveloping $n+k$ space. The first equation is that of a hypersphere, while the second set of equations of degree μ, $1 \leqq \mu \leqq n$, in general define a more complicated surface than a plane or quadratic surface.

The A's and B's of the preceding equation are determined in such a manner that the set of equations

$$\text{(polynomial)} = \varrho = \left(\frac{\partial x^i}{\partial u^\alpha}\right)^2 + \left(\frac{\partial x^{n+i}}{\partial u^\alpha}\right)^2,$$

$$0 = \frac{\partial x^i}{\partial u^a}\frac{\partial x^i}{\partial u^b} + \frac{\partial x^{n+j}}{\partial u^a}\frac{\partial x^{n+j}}{\partial u^b} \qquad \text{(summed on } i \text{ and } j\text{)}$$

are satisfied.

3. **Geodesic mapping.** We shall next consider the nature of the solution of the geodesics of any space. Let

$$\alpha^i = f^i\{\varphi^1(s), \cdots, \varphi^n(s)\} \qquad (i = 1, \cdots, n+k)$$

define the geodesics of V_{n+k}; then

$$\frac{dx^i}{ds} = \frac{\partial f^i}{\partial \varphi^l}\frac{d\varphi^l}{ds},$$

$$\frac{d^2x^i}{ds^2} = \frac{\partial^2 f^i}{\partial \varphi^l \partial \varphi^m}\frac{d\varphi^l}{ds}\frac{d\varphi^m}{ds} + \frac{\partial f^i}{\partial \varphi^l}\frac{d^2\varphi^l}{ds^2}.$$

If we set $\varphi^l = u^l$, then

$$\frac{d^2x^i}{ds^2} = \frac{\partial^2 f^i}{\partial u^l \partial u^m}\frac{du^l}{ds}\frac{du^m}{ds} + \frac{\partial f^i}{\partial u^l}\frac{d^2u^l}{ds^2}.$$

Now the equations of the geodesics in any space with the linear element

$$ds^2 = a_{\alpha\beta}\,du^\alpha\,du^\beta,$$

where

$$a_{\alpha\beta} = \frac{1}{n}f^a_\alpha f^a_\beta, \qquad f^a_\alpha = \frac{\partial f^a}{\partial u^\alpha},$$

are

$$\text{(I)} \qquad f^r_\alpha f^r_\beta \frac{d^2u^\alpha}{ds^2} + f^r_\beta f^r_{\alpha\delta}\frac{du^\alpha}{ds}\frac{du^\delta}{ds} = 0.$$

If the space V_{n+k} is Euclidean, then

$$\text{(II)} \qquad \frac{d^2x^i}{ds^2} = \frac{\partial^2 f^i}{\partial u^l \partial u^m}\frac{du^l}{ds}\frac{du^m}{ds} + \frac{\partial f^i}{\partial u^l}\frac{d^2u^l}{ds^2} = 0.$$

Multiplying this equation by $\partial f^i/\partial u^t$ and summing, we get

$$\frac{d^2x^i}{ds^2}\frac{\partial f^i}{\partial u^t} = 0.$$

Comparing I and II we see that a solution of the geodesic equation of any space determined by

$$a_{\alpha\beta} = \frac{\partial f^r}{\partial u^\alpha}\frac{\partial f^r}{\partial u^\beta} \qquad (\alpha, \beta = 1, \cdots, n)$$

is $u^\alpha = \varphi^\alpha(s)$, where φ^α is a solution of the equation

$$f^i(\varphi^1(s), \cdots, \varphi^n(s)) = a_i s + b_i.$$

We see, moreover, that any solution of

$$\frac{\partial^2 f^i}{\partial u^\alpha \partial u^\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} + \frac{\partial f^i}{\partial u^\alpha}\frac{d^2u^\alpha}{ds^2} = 0 \qquad (i = 1, \cdots, n+k)$$

satisfies the geodesic equation. This is useful in case $f^i = f^i(u^k)$ is a function of u^k alone; in this case

$$f^{i\prime\prime}\left(\frac{du^k}{ds}\right)^2 + f^{i\prime}\frac{d^2u^k}{ds^2} = 0,$$

which can be solved by quadratures.

The results proved above may be summarized as follows:

THEOREM. *The determination of the geodesics of any variety V_n' with the linear element*

$$ds^2 = a_{\alpha\beta}\,du^\alpha\,du^\beta$$

which correspond to the geodesics of the enveloping V_{n+k} that lie within the variety V_n, is reduced to an algebraic problem when we have found the solution of the system of differential equations

$$n\,a_{\alpha\beta} = \sum_{r=1}^{n+k} f_\alpha^r f_\beta^r.$$

We shall now compare the solution of the geodesic equations in two Euclidean spaces V_{n+k}' and E_{n+k}, where the linear element of V_{n+k}' is $ds^2 = a_{rs}'\,dy^r\,dy^s$ and that of E_{n+k} is $ds^2 = (dx^r)^2$. Then

$$\delta_{\alpha\beta} = a'_{rs}(y)\frac{\partial y^r}{\partial x^\alpha}\frac{\partial y^s}{\partial x^\beta} = a'_{rs}(\bar{f})\,\bar{f}^r_\alpha\,\bar{f}^r_\beta, \quad (r, s, \alpha, \beta = 1, \cdots, n+k)$$

where $\bar{f}$ is the reciprocal function to f, that is, $y^r = \bar{f}^r(x_1, \cdots, x_{n+k})$. The set of equations

$$a'_{rt}\frac{d^2y^r}{ds^2} + \Gamma'_{rs,t}\frac{dy^r}{ds}\frac{dy^s}{ds} = 0$$

becomes on transforming

$$a'_{rt}\left\{\frac{\partial^2\bar{f}^r}{\partial x^a\,\partial x^b}\frac{dx^a}{ds}\frac{dx^b}{ds} + \frac{\partial\bar{f}^r}{\partial x^a}\frac{d^2x^a}{ds^2}\right\} + \Gamma'_{rs,t}\frac{\partial\bar{f}^r}{\partial x^a}\frac{\partial\bar{f}^s}{\partial x^b}\frac{dx^a}{ds}\frac{dx^b}{ds} = 0.$$

But $x^j = a_j s + b$; hence

$$a'_{rt}\frac{\partial^2\bar{f}^r}{\partial s^2} + \Gamma'_{rs,t}\frac{\partial\bar{f}^r}{\partial s}\frac{\partial\bar{f}^s}{\partial s} = 0$$

and therefore $y^l = \bar{f}^l(\cdots, a^j s + b, \cdots)$ is a solution of the geodesic equation of V_{n+k}.

In the particular case $y^\alpha = u^\alpha$, $y^{n+l} =$ constant $(\alpha = 1, \cdots n;\ l = 1, \cdots, k)$, we have $u^\alpha = \bar{f}^\alpha(\cdots, a_j s + b, \cdots)$ and a_{n+l} and b_{n+l} must be determined so that $\bar{f}^{n+l}(\cdots, a_j s + b, \cdots)$ is constant.

ON CONTINUED FRACTIONS IN THE THEORY OF BINARY FORMS.

By A. Arwin.

In two previous papers* I have developed a theory of "reduced chains" by which, at least in its main lines, the theory of quadratic forms was followed up. I shall now prove that we may also use similar chains to solve analogous problems in the theory of binary cubic and higher forms. For this purpose I shall this time treat the following problem. Give the conditions by which an arbitrary number A may be expressed in the cubic form

$$b_0 x^3 + b_1 x^2 y + b_2 x y^2 + b_3 y^3 = A, \tag{1}$$

where b_0, b_1, b_2, b_3 are integers, and determine the solutions $(x, y) = (\alpha_i, \beta_i)$, $i = 1, 2, \cdots$, where α_i and β_i are integers. We shall see that the same method will always give the solutions even in the case of higher binary form when the fundamental equation (1), with $A = 0$, has at least one real root.

Let $x = \mu$ be a real root of

$$f(x) = b_0 x^3 + b_1 x^2 + b_2 x + b_3 = 0 \tag{2}$$

and $x = -P_\varrho$ a congruence-root of $f(x) \equiv 0\ (Q_\varrho)$, and form, as in the previous papers, a chain from the initial element $(\mu + P_\varrho)/Q_\varrho$ as follows:

$$\begin{aligned}
&\frac{\mu + P_\varrho}{Q_\varrho} = a_0 + \frac{\mu + \alpha_0}{Q_\varrho} = a_0 + \frac{1}{\frac{\alpha_1 \mu^2 + \beta_1 \mu + \gamma_1}{R_1}} = \omega_1, \\
&\frac{\alpha_1 \mu^2 + \beta_1 \mu + \gamma_1}{R_1} = a_1 + \frac{\alpha_1 \mu^2 + \beta_1 \mu + \gamma_1'}{R_1} = a_1 + \frac{1}{\frac{\alpha_2 \mu^2 + \beta_2 \mu + \gamma_2}{R_2}} = \omega_2, \\
&\qquad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots
\end{aligned} \tag{3}$$

where $a_0 = [(\mu + P_\varrho)/Q_\varrho]$, i. e., the greatest integer in $(\mu + P_\varrho)/Q_\varrho$, and where generally a_i is the greatest integer in ω_i.

We have first to prove some algebraic expressions in order to form the

* Periodically closed chains of reduced fractions, Ann. of Math. (2), vol. 24, pp. 39–68.
A contribution to the theory of closed chains, " " " (2) " 25, " 91–117.

chain (3). With $\omega_i = \dfrac{\alpha_i\mu^2+\beta_i\mu+\gamma_i}{R_i}$ and $\bar{\omega}_i$ as the associated number that is given by $\omega_i\bar{\omega}_i = 1$, we have

$$(4) \qquad (\alpha_i\mu^2+\beta_i\mu+\gamma_i)\,(\alpha_i'\mu^2+\beta_i'\mu+\gamma_i') = R_iR_i'.$$

Now, let ν and τ be the conjugate roots of μ; we obtain

$$\alpha_i'\mu^2+\beta_i'\mu+\gamma_i' = (\alpha_i\nu^2+\beta_i\nu+\gamma_i)\,(\alpha_i\tau^2+\beta_i\tau+\gamma_i),$$

and hence, by means of $\mu+\nu+\tau = -b_1/b_0 = -A_1$, $\mu\nu+\mu\tau+\nu\tau = b_2/b_0 = A_2$, $\mu\nu\tau = -b_3/b_0 = -A_3$, the following equations

$$\begin{aligned}
&\alpha_i' = \alpha_i^2A_2-\alpha_i\beta_iA_1+\beta_i^2-\alpha_i\gamma_i,\quad \beta_i = \alpha_i^2A_1A_2-\alpha_i^2A_3-\alpha_i\beta_iA_1^2+\beta_i^2A_1-\beta_i\gamma_i,\\
&\qquad \gamma_i' = (\gamma_i-\alpha_iA_2)^2+(\beta_i-\alpha_iA_1)\,(\beta_iA_2-\gamma_iA_1+\alpha_iA_3),\\
(5)\quad &R_iR_i' = \alpha_i'(-\beta_iA_3+\alpha_iA_1A_3)+\beta_i'(-\alpha_iA_3)+\gamma_i'\gamma_i\\
&\qquad = \alpha_i^3A_3^2-\alpha_i^2\beta_iA_2A_3+\alpha_i\beta_i^2A_1A_3+\alpha_i^2\gamma_i(A_2^2-2A_1A_3)\\
&\qquad\qquad +(3A_3-A_1A_2)\alpha_i\beta_i\gamma_i-\beta_i^3A_3+\alpha_i\gamma_i^2(A_1^2-2A_2)\\
&\qquad\qquad +\beta_i^2\gamma_iA_2-\beta_i\gamma_i^2A_1+\gamma_i^3.
\end{aligned}$$

Without any restriction in our methods and in order to simplify our formulas, we take $A_1 = 0$ and A_2, A_3 as integers, and have

$$\begin{aligned}
&\alpha_i' = \beta_i^2+\alpha_i^2A_2-\alpha_i\gamma_i, \qquad \beta_i' = -\alpha_i^2A_3-\beta_i\gamma_i,\\
(6)\qquad &\gamma_i' = (\gamma_i-\alpha_iA_2)^2+\beta_i(\alpha_iA_3+\beta_iA_2),\\
&R_iR_i' = \alpha_i'(-\beta_iA_3)+\beta_i'(-\alpha_iA_3)+\gamma_i'\gamma_i = \begin{vmatrix} \gamma_i-\alpha_iA_2 & \beta_i & \alpha_i\\ \alpha_iA_3+\beta_iA_2 & (\gamma_i-\alpha_iA_2) & -\beta_i\\ \beta_iA_3 & \alpha_iA_3 & -\gamma_i \end{vmatrix}.
\end{aligned}$$

We shall now prove the following theorem:

THEOREM 1. *Given any equivalence*

$$(7) \qquad \frac{\mu+P_\varrho}{Q_\varrho} = \frac{y_x\dfrac{\alpha_i\mu^2+\beta_i\mu+\gamma_i}{Q_\tau}+y_{x-1}}{z_x\dfrac{\alpha_i\mu^2+\beta_i\mu+\gamma_i}{Q_\tau}+z_{x-1}}, \qquad y_xz_{x-1}-y_{x-1}z_x = \pm 1,$$

where α_i and β_i as well as z_x and Q_τ are relatively prime, we conclude that

$$(8) \qquad \beta_i^3+\beta_i\alpha_i^2A_2+\alpha_i^3A_3 = \pm Q_\varrho Q_\tau.$$

In (7) the coefficients of μ^2 and μ give rise to the following relations

$$\begin{aligned} \beta_i z_x + \alpha_i z_x P_\varrho &= \alpha_i y_x Q_\varrho \\ -\alpha_i z_x A_2 + \gamma_i z_x + \beta_i z_x P_\varrho + z_{x-1} Q_\tau &= \beta_i y_x Q_\varrho \\ -\alpha_i z_x A_3 + \gamma_i z_x P_\varrho + z_{x-1} P_\varrho Q_\tau &= \gamma_i y_x Q_\varrho + y_{x-1} Q_\varrho Q_\tau . \end{aligned} \tag{9}$$

Assuming that α_i and β_i have the common factor s, so that $\alpha_i = \alpha_i' s$, $\beta_i = \beta_i' s$, where α_i' and β_i' are relatively prime, we take from the first equation in (9) $z_x \equiv 0\ (\alpha_i')$, $z_x = \alpha_i' t = z_x' t$, and, since y_x and z_x are relatively prime, we have also $Q_\varrho \equiv 0\ (t)$. Hence, since likewise z_{x-1}, z_x are relatively prime, the second equation of (9) gives $Q_\tau \equiv 0\ (t)$. Since $Q_\varrho = t Q_\varrho'$, $Q_\tau = t Q_\tau'$, (7) becomes

$$\frac{\mu + P_\varrho}{Q_\varrho'} = \frac{y_x \dfrac{\alpha_i \mu^2 + \beta_i \mu + \gamma_i}{Q_\tau'} + t y_{x-1}}{z_x' \dfrac{\alpha_i \mu^2 + \beta_i \mu + \gamma_i}{Q_\tau'} + z_{x-1}}, \qquad y_x z_{x-1} - y_{x-1} z_x' t = \pm 1. \tag{7'}$$

Hence

$$\begin{aligned} \beta_i' z_x' + \alpha_i' z_x' P_\varrho &= \alpha_i' y_x Q_\varrho' \\ -\alpha_i' s z_x' A_2 + \gamma_i z_x' + \beta_i' s z_x' P_\varrho + z_{x-1} Q_\tau' &= \beta_i' s y_x Q_\varrho' \\ -\alpha_i' s z_x' A_3 + \gamma_i z_x' P_\varrho + z_{x-1} P_\varrho Q_\tau' &= \gamma_i y_x Q_\varrho' + t y_{x-1} Q_\varrho' Q_\tau' . \end{aligned} \tag{9'}$$

Since now $z_x' = \alpha_i'$, we have from the second and third equations of (9')

$$\begin{aligned} \beta_i'^3 s + \beta_i' \alpha_i'^2 s A_2 &= \alpha_i' \beta_i' \gamma_i + \beta_i' z_{x-1} Q_\tau', \\ -\alpha_i'^3 s A_3 &= \alpha_i' \beta_i' \gamma_i + \alpha_i' t y_{x-1} Q_\varrho' Q_\tau' - \alpha_i' z_{x-1} P_\varrho Q_\tau', \end{aligned} \tag{9''}$$

that is

$$\begin{aligned} s(\beta_i'^3 + \beta_i' \alpha_i'^2 A_2 + \alpha_i'^3 A_3) &= z_{x-1}(\beta_i' Q_\tau' + \alpha_i' P_\varrho Q_\tau') - \alpha_i' t y_{x-1} Q_\varrho' Q_\tau' \\ &= (z_{x-1} y_x - z_x' y_{x-1} t) Q_\varrho' Q_\tau' = \pm Q_\varrho' Q_\tau' . \end{aligned} \tag{10}$$

Let s_τ be the greatest common factor of s and Q_τ'; we have $Q_\varrho' = s_\varrho Q_\varrho''$, $Q_\tau' = s_\tau Q_\tau''$, where s_ϱ and Q_τ'' are relatively prime. Hence

$$\beta_i'^3 + \beta_i' \alpha_i'^2 A_2 + \alpha_i'^3 A_3 = \pm Q_\varrho'' Q_\tau''. \tag{10'}$$

Further, since $\alpha_i' \gamma_i \equiv 0$, $\beta_i' \gamma_i \equiv 0\ (s_\tau)$, we can transform (7') into

$$\frac{\mu + P_\varrho}{s_\varrho Q_\varrho''} = \frac{y_x \dfrac{s_\varrho \alpha_i' \mu^2 + s_\varrho \beta_i' \mu + \gamma_i''}{Q_\tau''} + t y_{x-1}}{z_x' \dfrac{s_\varrho \alpha_i' \mu^2 + s_\varrho \beta_i' \mu + \gamma_i''}{Q_\tau''} + z_{x-1}}. \tag{7''}$$

But, with s_ϱ and Q''_τ relatively prime, the congruence $\gamma''_i - s_\varrho \gamma'_i \equiv 0\ (Q''_\tau)$ is soluble, and the equivalence (7″) can finally be written

$$(7''') \qquad \frac{\mu + P_\varrho}{Q''_\varrho} = \frac{s_\varrho\, y_x \dfrac{\alpha'_i \mu^2 + \beta'_i \mu + \gamma'_i}{Q''_\tau} + t\, y_{x-1} + \dfrac{\gamma''_i - s_\varrho \gamma'_i}{Q''_\tau} y_x}{z'_x \dfrac{\alpha'_i \mu^2 + \beta'_i \mu + \gamma'_i}{Q''_\tau} + \dfrac{z_{x-1} Q''_\tau + (\gamma''_i - s_\varrho \gamma'_i) z'_x}{s_\varrho\, Q''_\tau}},$$

where we must have $z_{x-1} Q''_\tau + z_x \gamma''_i \equiv 0\ (s_\varrho)$ because of (9″). Then we have transformed (7) into (7‴) of all the same type, where now α_i and β_i are relatively prime to each other as well as to Q_τ, and from (7′) we may conclude (8) in view of (10′). Conversely, we shall prove that from any relation (8) follows the equivalence (7). Before doing so we remark that (6) and (9) give rise to the congruences

$$(11) \quad \alpha'_i = \beta_i^2 + \alpha_i^2 A_2 - \alpha_i \gamma_i \equiv 0, \quad \beta'_i = -\alpha_i^2 A_3 - \beta_i \gamma_i \equiv 0. \quad (Q_\tau)$$

and, from these and (10′), $\gamma'_i \equiv 0\ (Q_\tau)$, relations to which we shall soon refer. Now, since

$$\beta_i^3 + \beta_i \alpha_i^2 A_2 + \alpha_i^3 A_3 = \pm Q_\varrho Q_\tau,$$

$$\alpha_i^3 \mu^3 + \alpha_i^3 \mu A_2 + \alpha_i^3 A_3 = 0,$$

we have

$$(\beta_i - \mu \alpha_i)(\beta_i^2 + \alpha_i \beta_i \mu + \alpha_i^2 \mu^2 + \alpha_i^2 A_2) = \pm Q_\varrho Q_\tau,$$

Since $-P_\varrho$ is a root of $x^3 + A_2 x + A_3 \equiv 0\ (Q_\varrho)$, for $\beta_i + \alpha_i P_\varrho \equiv 0\ (Q_\varrho)$, hence $\beta_i = y_x Q_\varrho - \alpha_i P_\varrho$, and we have

$$y_x - \alpha_i \frac{\mu + P_\varrho}{Q_\varrho} = \frac{\pm 1}{\alpha_i \dfrac{\alpha_i \mu^2 + \beta_i \mu + \gamma_i}{Q_\tau} + \dfrac{\beta_i^2 + \alpha_i^2 A_2 - \alpha_i \gamma_i}{Q_\tau}},$$

where γ_i is determined by $\beta_i^2 + \alpha_i^2 A_2 - \alpha_i \gamma_i \equiv 0\ (Q_\tau)$. We see that $\beta_i(\beta_i^2 + \alpha_i^2 A_2 - \alpha_i \gamma_i) + \alpha_i(\beta_i \gamma_i + \alpha_i^2 A_3) = \pm Q_\varrho Q_\tau$, hence $\beta_i \gamma_i + \alpha_i^2 A_3 \equiv 0\ (Q_\tau)$, and hence it is shown that the norm $N(\alpha_i \mu^2 + \beta_i \mu + \gamma_i)$ is $\equiv 0\ (Q_\tau)$. Putting $\omega_i = \dfrac{\alpha_i \mu^2 + \beta_i \mu + \gamma_i}{Q_\tau}$, $\alpha_i = z_i$ we have

$$\frac{\mu + P_\varrho}{Q_\varrho} = \frac{y_i \omega_i + \dfrac{y_i z_{i-1} \mp 1}{\alpha_i}}{z_i \omega_i + z_{i-1}},$$

and $y_i z_{i-1} \mp 1 \equiv 0\,(\alpha_i)$, since indeed $y_i z_{i-1} = \dfrac{\beta_i + \alpha_i P_\varrho}{Q_\varrho} \cdot \dfrac{\beta_i^2 + \alpha_i^2 A_2 - \alpha_i \gamma_i}{Q_\tau} \equiv$

$\equiv \dfrac{\beta_i^3}{Q_\varrho Q_\tau} \equiv \pm 1\,(\alpha_i)$. Hence the converse theorem is proved. Further, if μ_1 is the real root of the equation

$$x^m + A_2 x^{m-2} + A_3 x^{m-3} + \cdots + A_m = 0$$

and the continued fraction from μ_1 is formed so that

$$(11') \qquad \mu_1 = \frac{\beta_n \omega_n^{(1)} + \beta_{n-1}}{\alpha_n \omega_n^{(1)} + \alpha_{n-1}}, \qquad \beta_n \alpha_{n-1} - \beta_{n-1} \alpha_n = (-1)^n$$

where $\omega_n^{(1)} = (a_n^{(1)} \mu_1^{m-1} + a_n^{(2)} \mu_1^{m-2} + \cdots + a_n^{(m)})/R_n$, β_n and α_n are solutions in integers of $\beta_n^m + A_2 \beta_n^{m-2} + \cdots + A_m \alpha_n^m = (-1)^n R_n$, then from (11′) we have the following relation:

$$\begin{aligned} &\alpha_n a_n^{(1)} (-A_2 \mu_1^{m-2} - A_3 \mu_1^{m-3} - \cdots - A_m) \\ &\qquad + \alpha_n a_n^{(2)} \mu_1^{m-1} + \cdots + \alpha_n a_n^{(r)} \mu_1^{n-r-1} + \alpha_n a_n^{(m)} \mu_1 + R_n \alpha_{n-1} \mu_1 \\ &= \beta_n a_n^{(1)} \mu_1^{m-1} + \cdots + \beta_n a_n^{(r)} \mu_1^{m-r} + \cdots + a_n^{(m)} \beta_n + R_n \beta_{n-1}, \end{aligned}$$

that is, $\alpha_n a_n^{(2)} = \beta_n a_n^{(1)}$, $a_n^{(2)} = \lambda_1 \beta_n$, $a_n^{(1)} = \lambda_1 \alpha_n$, and further $\alpha_n a_n^{(3)} - \alpha_n a_n^{(1)} A_2 = \beta_n a_n^{(2)}$, hence $\alpha_n a_n^{(3)} = \lambda_1 (\beta_n^2 + \alpha_n^2 A_2)$, $a_n^{(3)} = \lambda_2 (\beta_n^2 + \alpha_n^2 A_2)$, that is, $a_n^{(3)} = \lambda_2 (\beta_n^2 + \alpha_n^2 A_2)$, $\lambda_1 = \lambda_2 \alpha_n$, or $a_n^{(1)} = \lambda_2 \alpha_n^2$, $\alpha_n^{(2)} = \lambda_2 \alpha_n \beta_n$. In the same way we find $\alpha_n a_n^{(4)} - \alpha_n a_n^{(1)} A_3 = \beta_n a_n^{(3)}$, $\alpha_n a_n^{(4)} = \lambda_2 (\alpha_n^3 A_3 + \beta_n \alpha_n^2 A_2 + \beta_n^3)$, $a_n^{(4)} = \lambda_3 (\beta_n^3 + A_2 \beta_n \alpha_n^2 + A_3 \alpha_n^3)$, $\lambda_2 = \lambda_3 \alpha_n$; $a_n^{(1)} = \lambda_3 \alpha_n^3$, $a_n^{(2)} = \lambda_3 \alpha_n^2 \beta_n$, $a_n^{(3)} = \lambda_3 \alpha_n (\beta_n^2 + \alpha_n^2 A_2)$, $a_n^{(4)} = \lambda_3 (\beta_n^3 + A_2 \beta_n \alpha_n^2 + A_3 \alpha_n^3)$; hence generally $a_n^{(1)} = \lambda_{r-1} \alpha_n^{r-1}$, $a_n^{(2)} = \lambda_{r-1} \alpha_n^{r-2} \beta_n$, $a_n^{(r)} = \lambda_{r-1} (\beta_n^{r-1} + A_2 \beta_n^{r-3} \alpha_n^2 + A_3 \beta_n^{r-4} \alpha_n^3 + \cdots + A_{r-1} \alpha_n^{r-1})$.

From the two equations

$$(11'') \qquad \begin{aligned} -\alpha_n a_n^{(1)} A_{m-1} + \alpha_n a_n^{(m)} + R_n \alpha_{n-1} - \beta_n a_n^{(m-1)} &= 0, \\ -\alpha_n a_n^{(1)} A_m \quad - a_n^{(m)} \beta_n - R_n \beta_{n-1} &= 0, \end{aligned}$$

we deduce

$$\begin{aligned} (-1)^n R_n &= a_n^{(1)} (A_{m-1} \alpha_n \beta_n + A_m \alpha_n^2) + \beta_n^2 a_n^{(m-1)} \\ &= \lambda_{m-2} (\beta_n^m + A_2 \beta_n^{m-2} \alpha_n^2 + A_3 \beta_n^{m-3} \alpha_n^3 + \cdots + A_{m-1} \beta_n \alpha_n^{m-1} + A_m \alpha_n^m); \end{aligned}$$

hence $R_n \equiv 0\,(\lambda_{m-2})$, that is, $a_n^{(m)} \equiv 0\,(\lambda_{m-2})$ or $\lambda_{m-2} = 1$, since $a_n^{(i)}$ and R_n are relatively prime.

It may also be remarked that even a quadratic irrationality ν, satisfying the equation $b_0x^2+b_1x+b_2=0$ as well as the reducible equation $(x-s)\cdot(b_0x^2+b_1x+b_2)=c_0x^3+c_1x^2+c_2x+c_3$, can be developed in a chain as a cubic irrationality, and proceeding from the cubic equation we attain a chain apparently different from the one from the quadratic equation. But the continued fraction for ν must in both cases be identically the same and hence periodic, though the fractions of the two chains are of somewhat different nature, as the following example shows.

$$\nu=1+\sqrt{13}. \qquad (x+2)(x^2-2x+12)=x^3-16x-24=0,$$

$$1+\sqrt{13}=4+\frac{\sqrt{13}-3}{1}=4+\frac{1}{\frac{\sqrt{13}+3}{4}},$$

$$\frac{\sqrt{13}+3}{4}=1+\frac{\sqrt{13}-1}{4}=1+\frac{1}{\frac{\sqrt{13}+1}{3}},$$

$$\frac{\sqrt{13}+1}{3}=1+\frac{\sqrt{13}-2}{3}=1+\frac{1}{\frac{\sqrt{13}+2}{3}},$$

$$\frac{\sqrt{13}+2}{3}=1+\frac{\sqrt{13}-1}{3}=1+\frac{1}{\frac{\sqrt{13}+1}{4}},$$

$$\nu=4+\frac{\nu-4}{1}=4+\frac{1}{\frac{\nu^2+4\nu+0}{24}=\frac{\sqrt{13}+3}{4}},$$

$$\frac{\nu^2+4\nu+0}{24}=1+\frac{1}{\frac{\nu^2+5\nu-12}{21}=\frac{\sqrt{13}+1}{3}},$$

$$\frac{\nu^2+5\nu-12}{21}=1+\frac{1}{\frac{2\nu^2+9\nu-11}{39}=\frac{\sqrt{13}+2}{3}},$$

$$\frac{2\nu^2+9\nu-11}{39}=1+\frac{1}{\frac{3\nu^2+14\nu-36}{80}=\frac{\sqrt{13}+1}{4}}.$$

. . . . . .

Let us now have the irreducible equation $f(x)=x^3+A_2x+A_3=0$ and consider $\mu>0$ as a real root of $f(x)=0$. If the chain for μ is formed as before, we attain the equivalence

$$(12) \qquad \mu = \frac{\beta_n \omega_n + \beta_{n-1}}{\alpha_n \omega_n + \alpha_{n-1}}, \qquad \beta_n \alpha_{n-1} - \beta_{n-1} \alpha_n = (-1)^n$$

hence

$$\mu - \frac{\beta_n}{\alpha_n} = \frac{(-1)^{n-1}}{\alpha_n^2 \left(\omega_n + \frac{\alpha_{n-1}}{\alpha_n}\right)} \to 0,$$

where $\omega_n = (\alpha_n \mu^2 + \beta_n \mu + \gamma_n)/R_n$ with β_n and α_n relatively prime to each other and R_n, which is easy to see from the formulas (9) for $P_0 = 0$, $Q_0 = 1$. Hence, and this is remarkable for these chains, we have without any exception

$$\beta_n^3 + A_2 \beta_n \alpha_n^2 + A_3 \alpha_n^3 = (-1)^n R_n, \quad \alpha_{n-1} = (\beta_n^2 + \alpha_n^2 A_2 - \alpha_n \gamma_n)/R_n = \alpha_n'/R_n,$$
$$\beta_{n-1} = (-\alpha_n^2 A_3 - \beta_n \gamma_n)/R_n = \beta_n'/R_n.$$

If γ_{n-1}' is the third coefficient of $\overline{\omega}_n = \omega_{n-1}^{(1)}$, $\omega_{n-1} = a_{n-1} + \omega_{n-1}^{(1)}$, we have

$$(\alpha_n \mu^2 + \beta_n \mu + \gamma_n)(\alpha_{n-1} \mu^2 + \beta_{n-1} \mu + \gamma_{n-1}') = R_{n-1} R_n,$$

and performing the multiplication we are led to the following relations

$$(13) \quad \begin{aligned} -\alpha_{n-1} \alpha_n A_2 + \beta_{n-1} \beta_n + \alpha_n \gamma_{n-1}' + \alpha_{n-1} \gamma_n &= 0, \\ -\alpha_{n-1} \alpha_n A_3 - \alpha_n \beta_{n-1} A_2 - \alpha_{n-1} \beta_n A_2 + \beta_n \gamma_{n-1}' + \beta_{n-1} \gamma_n &= 0, \\ -\alpha_n \beta_{n-1} A_3 - \alpha_{n-1} \beta_n A_3 + \gamma_{n-1}' \gamma_n &= R_{n-1} R_n. \end{aligned}$$

From the two former we take γ_n and γ_{n-1}' in expression of β_{n-1}, α_{n-1}, β_n, α_n as

$$(13') \quad \begin{aligned} (-1)^{n-1} \gamma_n &= \alpha_n^2 (\alpha_{n-1} A_3 + \beta_{n-1} A_2) + \beta_n^2 \beta_{n-1}; \\ (-1)^n \gamma_{n-1}' &= \alpha_{n-1}^2 (\alpha_n A_3 + \beta_n A_2) + \beta_{n-1}^2 \beta_n; \\ (-1)^n R_n &= \beta_n^3 + A_2 \beta_n \alpha_n^2 + A_3 \alpha_n^3. \end{aligned}$$

From the theoretical point of view these formulas and even more the already found

$$(9) \quad \begin{aligned} R_n \alpha_{n-1} &= \alpha_n^2 A_2 + \beta_n^2 - \alpha_n \gamma_n, \\ R_n \beta_{n-1} &= -\alpha_n^2 A_3 - \beta_n \gamma_n, \\ R_n \gamma_{n-1}' &= (\gamma_n - \alpha_n A_2)^2 + \beta_n (\alpha_n A_3 + \beta_n A_2), \end{aligned}$$

as well as the associated

$$
\begin{aligned}
R_{n-1}\alpha_n &= \alpha_{n-1}^2 A_2 + \beta_{n-1}^2 - \alpha_{n-1}\gamma'_{n-1},\\
R_{n-1}\beta_n &= -\alpha_{n-1}^2 A_3 - \beta_{n-1}\gamma'_{n-1},\\
R_{n-1}\gamma_n &= (\gamma'_{n-1} - \alpha_{n-1}A_2)^2 + \beta_{n-1}(\alpha_{n-1}A_3 + \beta_{n-1}A_2),
\end{aligned}
\tag{11}
$$

are of great importance; but in forming the chain the following expressions are preferable

$$
\alpha_{n+1} = a_n\alpha_n + \alpha_{n-1}, \qquad \beta_{n+1} = a_n\beta_n + \beta_{n-1}, \tag{13''}
$$

where a_i are the coefficients of the continued fraction. Changing n in $n+1$ in (13′) and substituting in (13″), we obtain two linear recursion-formulas for γ_{n+1} and R_{n+1}

$$
\begin{aligned}
\gamma_{n+1} &= a_n^2 R_n + 2a_n A_2\alpha_n - 2a_n\gamma_n + \gamma'_{n-1}\\
R_{n+1} &= R_{n-1} - a_n\gamma_{n+1} + a_n^2\gamma_n - 2a_n\gamma'_{n-1} + 2a_n A_2\alpha_{n-1}.
\end{aligned}
\tag{13'''}
$$

I shall for the present merely indicate the discussion of these formulas reserving a fuller treatment for a subsequent occasion. Calling the conjugate roots of $\omega_n\,\omega'_n$ and ω''_n we readily compute $\omega_n + \omega'_n + \omega''_n = (3\gamma_n - 2A_2\alpha_n)/R_n$, $\omega_n\omega'_n\omega''_n = R_{n-1}/R_n$, $\omega_n\omega'_n + \omega_n\omega''_n + \omega'_n\omega''_n = \omega_n\omega'_n\omega''_n\left(\frac{1}{\omega_n} + \frac{1}{\omega'_n} + \frac{1}{\omega''_n}\right)$ $= (3\gamma'_{n-1} - 2A_2\alpha_{n-1})/R_n$ hence the irreducible equation for ω_n is

$$
\psi_n(x) = R_n x^3 - (3\gamma_n - 2A_2\alpha_n)x^2 + (3\gamma'_{n-1} - 2A_2\alpha_{n-1})x - R_{n-1} = 0.
$$

But, since $\omega_{n-1} = a_{n-1} + \frac{1}{\omega_n}$,

$$
\begin{aligned}
0 = \psi_{n-1}(\omega_{n-1}) = \psi_{n-1}\left(a_{n-1} + \frac{1}{\omega_n}\right) &= \psi_{n-1}(a_{n-1}) + \frac{1}{\omega_n}\psi'_{n-1}(a_{n-1})\\
&+ \frac{1}{\omega_n^2}\cdot\frac{1}{2!}\psi''_{n-1}(a_{n-1}) + \frac{1}{\omega_n^3}\cdot\frac{1}{3!}\psi'''_{n-1}(a_{n-1}),
\end{aligned}
$$

that is, ω_n satisfies also

$$
\psi_{n-1}(a_{n-1})x^3 + \psi'_{n-1}(a_{n-1})x^2 + \frac{1}{2!}\psi''_{n-1}(a_{n-1})x + \frac{1}{3!}\psi'''_{n-1}(a_{n-1}) = 0.
$$

It is readily verified that $\frac{1}{3!}\psi'''_{n-1}(a_{n-1}) = R_{n-1}$, hence the equalities

$\psi_{n-1}(a_{n-1}) = -R_n$, $\psi'_{n-1}(a_{n-1}) = 3\gamma_n - 2A_2\alpha_n$, $-\frac{1}{2!}\psi''_{n-1}(a_{n-1}) = 3\gamma'_{n-1} - 2A_2\alpha_{n-1}$. Without following up this discussion of the equations $\psi_n(x) = 0$ we only remark that formerly the development of a cubic irrationality in a regular continued fraction has indeed employed this set of equations $\psi_n(x) = 0$ for determining the coefficients a_i.

We shall also note the following relation. Multiplying (13′) by α_{n-1}/α_n^2 and subtracting the analogous equation of R_{n-1}, multiplied by α_n/α_{n-1}^2 we have

$$\frac{R_n\alpha_{n-1}}{\alpha_n^2} + \frac{R_{n-1}\alpha_n}{\alpha_{n-1}^2} = \left(\frac{\beta_n}{\alpha_n}\right)^2 + \frac{\beta_n}{\alpha_n}\cdot\frac{\beta_{n-1}}{\alpha_{n-1}} + \left(\frac{\beta_{n-1}}{\alpha_{n-1}}\right)^2 + A_2 \to f'(\mu),$$

and also from

$$\left(\frac{\beta_n}{\alpha_n}\right)^3 + A_2\left(\frac{\beta_n}{\alpha_n}\right) + A_3 = \frac{(-1)^n R_n}{\alpha_n^3}, \qquad \mu^3 + A_2\mu + A_3 = 0$$

the relation

$$\left(\mu - \frac{\beta_n}{\alpha_n}\right)\left(\mu^2 + \mu\frac{\beta_n}{\alpha_n} + \left(\frac{\beta_n}{\alpha_n}\right)^2 + A_2\right) = \frac{(-1)^{n-1}R_n}{\alpha_n^3},$$

that is,

$$\left|\mu - \frac{\beta_n}{\alpha_n}\right| \to \frac{R_n}{\alpha_n^3 f'(\mu)}. \tag{12′}$$

THEOREM 2. *Any equation*

$$y^3 + A_2 yx^2 + A_3 x^3 = R \tag{11‴}$$

can, for given R, only afford a finite number of solutions in integers (x_i, y_i).

This theorem is proved in general by Mr. Thue* in virtue of the following lemma: If μ is a real root of an algebraic equation of degree r, the inequality

$$|y - \mu x| < c/x^{\frac{r}{2}+k},$$

where c and k are arbitrary constants, can only be satisfied by a finite number of solutions.

Referring to our formulas above we deduce

$$\omega_n + \frac{\alpha_{n-1}}{\alpha_n} = \frac{\alpha_n}{R_n}\left(\left(\frac{\beta_n}{\alpha_n}\right)^2 + \frac{\beta_n}{\alpha_n}\mu + \mu^2 + A_2\right) = \frac{S_n\alpha_n}{R_n} \to \frac{\alpha_n}{R_n}f'(\mu), \tag{12‴}$$

* Thue, A., Journ. für Math., Bd. 135, S. 284.

where $\alpha_n > \alpha_{n-1}$ and $S_n \to f'(\mu)$ for $n \to \infty$. Substituting $(\mu : \mu_2)$ in ω_n we have $\omega_n' = \frac{\alpha_n}{R_n}\left(\mu_2^2 + \frac{\beta_n}{\alpha_n}\mu_2 + \frac{\gamma_n}{\alpha_n}\right)$ and from (12′)

$$\omega_n' + \frac{\alpha_{n-1}}{\alpha_n} = \frac{(-1)^n}{\alpha_n^2\left(\mu_2 - \frac{\beta_n}{\alpha_n}\right)} \to 0, \tag{12″}$$

whence in all cases ω_n' must be limited and $\omega_n' \to \omega_n'' \to -\frac{\alpha_{n-1}}{\alpha_n}$. We can distinguish three cases: ω_n limited, and from (12‴) we conclude $R_n \to \infty$; $\omega_n \to \infty$ for all n without any exception, and this time also it is possible to show $R_n \to \infty$, the proof of which however we omit here; und lastly, the possibility $\omega_{n_1} \to \infty$ only for a set of values n_1 in n with R_{n_1} limited in these ω_{n_1}. I refer to the above theorem of Mr. Thue to show that this case is to be excluded; for we should come to relations (12′) with $R_{n_1} = R$ for an infinite number of indices n_1, contrary to the inequality of Mr. Thue. Probably these results can be deduced also from our point of view, and I shall try to give a proof of this in the future. Hence in all cases we must have $R_n \to \infty$ in ω_n; but this fact being granted, it is readily proved that Theorem 2 is true for all $f_m(x) = 0$, reducible or not. We take the irreducible $f_3(x) = 0$ and show under the above assumptions that the diophantine equation

$$\beta_i^3 + A_2\beta_i\alpha_i^2 + A_3\alpha_i^3 = \pm R \tag{13′}$$

cannot have an infinite number of solutions. We deduce, as above, $\mu - \beta_i/\alpha_i \to \frac{\mp R}{\alpha_i^3 f'(\mu)}$ and obtain $R/\alpha_i f'(\mu) < \frac{1}{2}$ for $\alpha_i > 2R/f'(\mu)$, hence β_i/α_i must be a convergent of the continued fraction from μ.* But then $\alpha_i/R \to \infty$ against the assumption that all α_n/R_n are limited or $R_n \to \infty$. We may however conclude, and this is important, that all finite solutions of (13′) of more considerable value must be attained by the regular formation of chain.

If α_n/R_n are limited, they have on the finite part of the real axis a finite, perhaps an infinite, number of points of condensation. The inferior limit is, as we readily see, > 0 and the superior $< \chi$, if χ is finite. We must then have $\alpha_n/R_n < \chi + 1$ or $R_n > \alpha_n/(\chi+1)$ for $n > N$, and this is a remarkable inequality. For, let us suppose that we seek the greatest solution of (13′) and that we have determined χ, then from $R_n > \alpha_n/(\chi+1)$ we can infer that, having reached an α_τ with $R \leqq \alpha_\tau/(\chi+1)$, there are

* Perron, O., Kettenbrüche, S. 45, Satz II.

no more solutions to be found by the formation of the chain. This problem, to determine in practice a limit $k(\mu)$, when existing, independent of n, such that $R_n > k(\mu)\,\alpha_n$ for $n > N$, is interesting and, as it seems, not easy.

Another question of interest is to decide the signs of γ'_{n-1} and γ_n in $\bar{\omega}_n$ and ω_n. In the special case of three real roots with two positive, that is with $A_3 > 0$, it is readily done for the two positive ones, but it is in fact not so easy in all cases. From (13) we have

$$\frac{\gamma'_{n-1}}{\alpha_{n-1}} + \frac{\gamma_n}{\alpha_n} \to -\mu^2 + A_2, \qquad \frac{\gamma'_{n-1}}{\alpha_{n-1}}\,\frac{\gamma_n}{\alpha_n} \to \frac{R_{n-1}}{\alpha_{n-1}}\,\frac{R_n}{\alpha_n} + 2A_3\mu.$$

Supposing $[\mu_2] = a_0 > \mu_3$ it can be proved that all α_n and β_n in the fractions from μ_2 are positive, which does not happen in the chain from μ_3, as we shall see in examples below. Hence from the second equation we infer that γ'_{n-1} and γ_n hold the same sign and from the first, since $A_2 < 0$, this sign is negative.

Let us now suppose that μ_1 is a real root of the equation $f_m(x) = 0$ and that $\omega_n^{(1)}$ is possibly limited for all n or at least $R_n \to \infty$. Thus we have $\mu_1 - \dfrac{\beta_n}{\alpha_n} = \dfrac{(-1)^{n-1}}{\alpha_n^2\left(\omega_n^{(1)} + \dfrac{\alpha_{n-1}}{\alpha_n}\right)}$ and also $\mu_\tau - \dfrac{\beta_n}{\alpha_n} = \dfrac{(-1)^{n-1}}{\alpha_n^2\left(\omega_n^{(\tau)} + \dfrac{\alpha_{n-1}}{\alpha_n}\right)}$ for all conjugate values of μ_1. Since $\mu_1 - \dfrac{\beta_n}{\alpha_n} \to 0$ there necessarily follows $\omega_n^{(\tau)} + \dfrac{\alpha_{n-1}}{\alpha_n} \to 0$ for all $\tau \neq 1$, as already in equation (12''). Further, there exists a discriminant relation $R_n^{2(m-1)}\,\Delta(\omega_n^{(1)}) = \Delta(\mu_1)$, where $\omega_n^{(1)} = (a_n^{(1)}\mu_1^{m-1} + a_n^{(2)}\mu_1^{m-2} + \ldots + a_n^{(m)})/R_n$, verified by (12) and $(-1)^n R_n = \prod_{(\tau)}(\beta_n - \mu_\tau\,\alpha_n) = \dfrac{(-1)^n}{\prod_{(\tau)}(\alpha_n\,\omega_n^{(\tau)} + \alpha_{n-1})}$, whence $R_n \to \infty$, since $\omega_n^{(i)} - \omega_n^{(\tau)} \to 0$ for $i \neq \tau \neq 1$ if $\omega_n^{(1)}$ itself is limited; but this can be directly concluded from the assumption that $\omega_n^{(1)}$ is limited since $\mu_1 - \dfrac{\beta_n}{\alpha_n} \to \dfrac{(-1)^{n-1}R_n}{\alpha_n^m f'_m(\mu_1)}$, and hence $\omega_n^{(1)} + \dfrac{\alpha_{n-1}}{\alpha_n} \to \dfrac{\alpha_n^{m-2} f'_m(\mu_1)}{R_n}$, $\alpha_n \to \infty$. And in the same way as previously we infer that the diophantine equation

$$y^m + A_2\,y^{m-2}\,x^2 + A_3\,y^{m-3}\,x^3 + \ldots + A_m\,x^m = \pm R \tag{14}$$

has only a finite number of solutions in integers (β_i, α_i) $i = 1, 2 \cdots \tau$. When μ_1 satisfies $x^3 + A_2 x + A_3 = 0$, we readily verify

$$\omega_n^{(2)} - \omega_n^{(3)} = \frac{\alpha_n}{R_n}(\mu_2 - \mu_3)\left(\mu_2 + \mu_3 + \frac{\beta_n}{\alpha_n}\right) \to 0$$

and

$$\omega_n^{(1)} - \omega_n^{(2)} = \frac{\alpha_n}{R_n}(\mu_1 - \mu_2)\left(\mu_1 + \mu_2 + \frac{\beta_n}{\alpha_n}\right)$$

$$= \frac{\alpha_n}{R_n}(\mu_1 - \mu_2)\left(\frac{\beta_n}{\alpha_n} - \mu_3\right) \to \frac{\alpha_n}{R_n}(\mu_1 - \mu_2)(2\mu_1 + \mu_2)$$

where $(\mu_1 - \mu_2)(2\mu_1 + \mu_2) = 2\mu_1^2 - \mu_1\mu_2 - \mu_2^2 = f'(\mu_1)$, since $\mu_1^2 + \mu_1\mu_2 + \mu_2^2 + A_2 = 0$. Further from $\omega_n^{(1)} - \omega_n^{(2)} = \frac{\alpha_n}{R_n}(\mu_1 - \mu_2)\left(\frac{\beta_n}{\alpha_n} - \mu_3\right)$ and $R_n^4 \Delta(\omega_n) = \Delta(\mu_1)$ we have $R_n^4 \left(\frac{\alpha_n}{R_n}\right)^6 \left(\frac{\beta_n^3 + A_2\beta_n\alpha_n^2 + A_3\alpha_n^3}{\alpha_n^3}\right)^2 \Delta(\mu_1) = \Delta(\mu_1)$, which gives $\beta_n^3 + A_2\beta_n\alpha_n^2 + A_3\alpha_n^3 = \pm R_n$ once more. Analogously we get from $x^4 + A_2 x^3 + A_3 x^3 + A_4 = 0$

$$\omega_n^{(1)} - \omega_n^{(2)} = \frac{\alpha_n^2}{R_n}(\mu_1 - \mu_2)\left(\mu_1^2 + \mu_1\mu_2 + \mu_2^2 + \frac{\beta_n}{\alpha_n}(\mu_1 + \mu_2) + \left(\frac{\beta_n}{\alpha_n}\right)^2 + A_2\right) =$$

$$= \frac{\alpha_n^2}{R_n}(\mu_1 - \mu_2)\left(\frac{\beta_n}{\alpha_n} - \mu_3\right)\left(\frac{\beta_n}{\alpha_n} - \mu_4\right),$$

since $\mu_1 + \mu_2 = -(\mu_3 + \mu_4)$, $A_2 = \sum \mu_i\mu_\tau = \mu_1(\mu_2 + \mu_3 + \mu_4) + \mu_2(\mu_1 + \mu_3 + \mu_4) - \mu_1\mu_2 + \mu_3\mu_4$, and also $\omega_n^{(i)} - \omega_n^{(\tau)} \to 0$ for $i \neq \tau \neq 1$, thus from $R_n^6 \Delta(\omega_n) = \Delta(\mu_1)$

$$R_n^6 \Delta(\mu_1)\left(\frac{\alpha_n^2}{R_n}\right)^{12}\left(\frac{\beta_n^4 + A_2\beta_n^2\alpha_n^2 + A_3\beta_n\alpha_n^3 + A_4\alpha_n^4}{\alpha_n^4}\right)^6 = \Delta(\mu_1),$$

hence the relation (14).

Finally, if μ_1 satisfies any quadratic equation $f_2(x) = A_0x^2 + 2A_1x + A_2 = 0$, we have formerly proved that the chain from μ_1 is periodically closed and hence the diophantine equations

$$A_0\beta_{i+\tau}^2 + 2A_1\beta_{i+\tau}\alpha_{i+\tau} + A_3\alpha_{i+\tau}^2 = (-1)^{i+\tau}R_i \quad (i = 1, 2 \cdots \tau),$$

where τ is the period, have an infinite number of solutions. Hence

$$\mu_1 - \frac{\beta_n}{\alpha_n} \to \frac{(-1)^{n-1}R_\varrho}{\alpha_n^2 f_2'(\mu_1)},$$

from which we conclude that β_n/α_n for $n \to \infty$ approximates μ_1 as $R_\varrho/2\sqrt{D}\alpha_n^2$, since $f_2'(\mu_1) = 2\sqrt{D}$, and that we reach the best approximation with

$R_\varrho = 1$ and also the set of values Q_n in $\mu - \frac{\beta_n}{\alpha_n} = \frac{Q_n}{\alpha_n^2}$ for $n \to \infty$ must have a finite set of points of condensation $R_\varrho / 2\sqrt{D}$, $\varrho = 1, 2 \cdots \tau$, cyclically arranged. Let $\omega_n^{(1)}$ satisfy the algebraic equation $\psi_n(x) = A_0^{(n)} x^m + A_1^{(n)} x^{m-1} + \cdots + A_m^{(n)} = 0$; then, since $\omega_n^{(i)} - \omega_n^{(\tau)} \to 0$ for all $i \neq \tau \neq 1$, we can approximately put $\omega_n^{(i)} \sim \omega_n^{(2)}$, $i = 3 \cdots m$, and have $\psi_n^r(\omega_n^{(2)}) \sim 0$, $r = 1, 2 \cdots (m-2)$, but $\psi_n^{m-1}(\omega_n^{(2)}) \sim 0$. Hence $m(m-1) \cdots \cdots 3 A_0^{(n)} (\omega_n^{(2)})^2 + (m-1) \cdots 2 A_1^{(n)} \omega_n^{(2)} + (m-2) \cdots 2 \cdot 1 A_2^{(n)} \sim 0$, but we have also $\omega_n^{(1)} + (m-1)\omega_n^{(2)} \sim -A_1^{(n)}/A_0^{(n)}$, hence $m A_0^{(n)} (\omega_n^{(1)} + A_1^{(n)}/A_0^{(n)})^2 - - 2(m-1) A_1^{(n)} (\omega_n^{(1)} + A_1^{(n)}/A_0^{(n)}) + 2(m-1) A_2^{(n)} \sim 0$. The equation $\varrho_n(x) = x^2 + 2A_1^{(n)}/A_0^{(n)} m \cdot x + [2(m-1)/m \cdot A_2^{(n)}/A_0^{(n)} - (m-2)/m \cdot \cdot (A_1^{(n)}/A_0^{(n)})^2] = 0$ has for increasing n the real root $\varrho_n^{(1)} = (-A_1^{(n)} + + \sqrt{(m-1)[(m-1)(A_1^{(n)})^2 - 2A_2^{(n)} A_0^{(n)} m)}]/A_n^{(1)} m$, and we see, since $\varrho_n^{(1)}$ approximates $\omega_n^{(1)}$, that $\omega_n^{(1)}$ will follow $\varrho_n^{(1)}$ in the chain, say, in m_1 coefficients a_i, and $\omega_{n+m_1}^{(1)}$ will follow $\varrho_{n+m_1}^{(1)}$, say, m_2 step, and so on with increasing accuracy. Then according as $\varrho_n^{(1)}$ by $n \to \infty$ has a finite period or not, possibly the period is under circumstances at least finite, if ω_n is* limited. The set $R_n / f'(\mu_1)\, \alpha_n^{m-2}$ will converge to a finite number of cyclically arranged points of condensation or not.

It is to be remarked that this procedure of forming chains and most of the conclusions from it is employable even on complex roots.

We now return to the cubic irrationalities and treat in greater detail the problem of finding all solutions in integers of the equation

$$y^3 + A_2 y x^2 + A_3 x^3 = \pm R. \tag{1'}$$

We have obviously: The necessary and sufficient condition for solutions of (1') is the existence of an equivalence (7), where $Q_\varrho = R$, $Q_\tau = 1$ or $Q_\varrho = 1$, $Q_\tau = R$, and we reach all solutions, finite in number, on forming the equivalences

$$\frac{\mu + P_\varrho}{R} = \frac{y_x \omega + y_{x-1}}{z_x \omega + z_{x-1}} \qquad y_x z_{x-1} - y_{x-1} z_x = \pm 1, \tag{15}$$

where $\omega = \alpha\mu^2 + \beta\mu - \alpha\gamma$, $z_{x-1} = \beta^2 + \alpha^2 A_2 - \alpha\gamma$, $z_x = \alpha$ and where γ is uniquely determined by the condition $z_{x-1} > 0$, $z_{x-1} = \beta^2 + \alpha^2 A_2 - \alpha\gamma < z_x = \alpha$ or $(\beta^2 + \alpha^2 A_2)/\alpha - \gamma < 1$. For $\beta^2 + \alpha^2 A_2 > 0$ we have $\gamma = \left[\frac{\beta^2 + \alpha^2 A_2}{\alpha}\right] > 0$, and for $\beta^2 + \alpha^2 A^2 < 0$ we take $\gamma < 0$ and $|\gamma| = \left[\frac{\beta^2 + \alpha^2 A_2}{\alpha}\right] + 1$, thus $z_{x-1} < z_x$ and from $y_x z_{x-1} - y_{x-1} z_x = \pm 1$ there follows that $y_x z_{x-1}/z_x = y_{x-1} \pm 1/z_x$, where $z_x \geqq 2$ with $x > 2$, hence $y_x > y_{x-1} \pm 1/z_x$,

* as for example in the reducible equation p. 252.

that is $y_x > y_{x-1}$ always in the case of the sign $+$ and $y_x \geqq y_{x-1}$ in the case of $-$, but since y_x and y_{x-1} are relatively prime, we must have $y_x > y_{x-1}$, $y_x = y_{x-1} = 1$ excepted. Having computed γ, ω is determined and either (1) $\omega > 0$ or (2) $\omega < 0$. In the first case we distinguish (1′) $\omega > 1$ and (1″) $\omega < 1$, and for (1′) we calculate $y_x/z_x = a_0 + \frac{1}{|a_1|} + \cdots + \frac{1}{|a_x|}$. Let us set $\omega = \omega_{x+1}$, $\omega_x = a_x + 1/\omega_{x+1}$. If now $\omega_{x+1} > 1$, a_x is the greatest integer in ω_x, and generally a_i the greatest number in ω_i for all i; hence ultimately a_0 must be the greatest integer in $(\mu + P_\varrho)/R_\varrho$. Hence under the conditions (1′) we can deduce the equivalence (14) simply by forming the regular chain, proceeding from the given element $(\mu + P_\varrho)/R_\varrho$. But this is no longer possible if $\omega < 1$. If $y_x z_{x-1} - y_{x-1} z_x = +1$, we can readily reduce this case to the preceding as follows. We write

$$(15') \qquad \frac{\mu + P_\varrho}{R_\varrho} = \frac{y_x(\omega+1) - (y_x - y_{x-1})}{z_x(\omega+1) - (z_x - z_{x-1})}$$

with $y_x(z_x - z_{x-1}) - z_x(y_x - y_{x-1}) = -1$, $y_x > y_x - y_{x-1} > 0$, $z_x > z_x - z_{x-1} > 0$, whence in $y_x/z_x = b_0 - \frac{1}{|b_1|} - \cdots - \frac{1}{|b_x|}$ we have $(y_x - y_{x-1})/(z_x - z_{x-1}) = b_0 - \frac{1}{|b_1|} - \cdots - \frac{1}{|b_{x-1}|}$; hence we must attain (15′) by forming from $(\mu + P_\varrho)/R_\varrho$ the second type of regular chains, in which $y_\tau z_{\tau-1} - y_{\tau-1} z_\tau = -1$ for all τ. Also in the case $\omega < 0$, $|\omega| > 1$ and $y_x z_{x-1} - y_{x-1} z_x = -1$ we may proceed as above. But there are also cases where the equivalence (15) cannot be deduced by any process of regular formation of chain, and consequently there would exist solutions of (1) which could not be reached by this method. We have, however, already seen that all solutions (β_i, α_i) which surpass a given limit $i = N$ must be reached by the regular formation of chain; and we shall see that there exist cubic forms and values R for which we have no exceptional solutions, i. e., solutions not to be attained by a regular formation of chain. Further we may remark that the congruence $x^3 + A_2 x + A_3 \equiv 0 \ (R)$* can have three different solutions P_ϱ, but that only one or two of them afford solutions (15). If, however, several solutions are deduced from the same chain, they are necessarily connected by some unit in much the same way as in the theory of quadratic forms.

THEOREM 4. *If from the same initial element several ω_i having the same denominator be reached, these ω_i can differ only by unit factors.*

* Concerning solutions of congruences see, Arwin, A., Über Kongruenzen von dem fünften und höheren Graden nach einem Primzahlmodulus, Arkiv för Mat., Astron. och Fysik, Bd. 14, 1918, no. 7.

Let us therefore suppose that (β, α) and (β_1, α_1) are two different solutions of (1'). Equation (9) yields $\beta = Ry_x - P_\varrho z_x$, $z_x = \alpha$ and $\beta_1 = Ry'_x - P_\varrho z'_x$, $z'_x = \alpha_1$, whence $\beta_1\alpha - \beta\alpha_1 \equiv 0\ (R)$, $\beta_1\alpha - \beta\alpha_1 = Rm$. Further, from

$$(\beta - \alpha\mu)(\beta_1^2 + \beta_1\alpha_1\mu + \alpha_1^2\mu^2 + \alpha_1^2 A_2) = \beta\beta_1^2 + \beta\alpha_1\beta_1\mu + \beta\alpha_1^2\mu^2 + \beta\alpha_1^2 A_2 -$$
$$- \alpha\beta_1^2\mu - \alpha\alpha_1\beta_1\mu^2 + \alpha\alpha_1^2(A_2\mu + A_3) - \alpha\alpha_1^2 A_2\mu = \mu^2\alpha_1(\alpha_1\beta - \alpha\beta_1) +$$
$$+ \mu\beta_1(\alpha_1\beta - \alpha\beta_1) + \beta(\beta_1^2 + \alpha_1^2 A_2) + \alpha\alpha_1^2 A_3.$$

Since $\beta_1(\beta_1^2 + \alpha_1^2 A_2) + \alpha_1^3 A_3 \equiv 0$, $\beta_1\alpha \equiv \beta\alpha_1\ (R)$, we have $\beta(\beta_1^2 + \alpha_1^2 A_2) + \alpha\alpha_1^2 A_3 = Rs$, where s is an integer. Consequently $-m\alpha_1\mu^2 - m\beta_1\mu + s = E$, just precisely the unit in Theorem 4.

Example. $x^3 - 5 = 0$, $\mu \sim 1{,}71$, $\mu^2 \sim 2{,}92$.

$$x^3 - 5 \equiv (x+2)(x+5)(x+6) \equiv 0\ (13),$$

$$\frac{\mu+2}{13} = 0 + \frac{\mu+2}{13} = 0 + \cfrac{1}{\cfrac{\mu^2-2\mu+4}{1}},$$

$$\frac{\mu^2-2\mu+4}{1} = 3 + \frac{\mu^2-2\mu+1}{1} = 3 + \cfrac{1}{\cfrac{3\mu^2+7\mu+11}{16}},$$

$$\frac{3\mu^2+7\mu+11}{16} = 1 + \frac{3\mu^2+7\mu-5}{16} = 1 + \cfrac{1}{\cfrac{4\mu^2+5\mu-5}{15}},$$

$$\frac{4\mu^2+5\mu-5}{15} = 1 + \frac{4\mu^2+5\mu-20}{15} = 1 + \cfrac{1}{\cfrac{7\mu^2+12\mu+20}{1}}.$$

Thus we have immediately the two solutions $(-2)^3 - 5\cdot 1^3 = -13$ and $12^3 - 5\cdot 7^3 = 13$; hence $12\cdot 1 - 7\ (-2) = 13\cdot 2$, that is, $m = 2$ and $12\cdot 4 - 7\cdot 5 = 13\cdot 1$, and consequently $2\mu^2 - 4\mu + 1$ is the unit by which the two solutions are connected, and in this case the fundamental unit in $K(\sqrt[3]{5})$. From $(\mu+5)/13$ we have

$$\frac{\mu+5}{13} = 0 + \frac{\mu+5}{13} = 0 + \cfrac{1}{\cfrac{\mu^2-5\mu+25}{10}},$$

$$\frac{\mu^2-5\mu+25}{10} = 1 + \frac{\mu^2-5\mu+15}{10} = 1 + \cfrac{1}{\cfrac{\mu^2+8\mu+25}{39}},$$

$$\frac{\mu^2+8\mu+25}{39} = 1 + \frac{\mu^2+8\mu-14}{39} = 1 + \cfrac{1}{\cfrac{2\mu^2+3\mu+4}{1}},$$

. . . . . .

and in fact the solution $3^3 - 5 \cdot 2^3 = -13$ belongs to the congruence-root -5. The element $(\mu+6)/13$ does not lead to any solution.

Example. $x^3 - 6x - 7 = 0$, $\mu \sim 2{,}91$, $R = 11$,

$$(x-2)(x+6)(x+7) \equiv 0 \,(11),$$

$$\frac{\mu+7}{11} = 0 + \frac{\mu+7}{11} = 0 + \frac{1}{\frac{\mu^2-7\mu+43}{27}},$$

$$\frac{\mu^2-7\mu+43}{28} = 1 + \frac{\mu^2+7\mu+15}{28} = 1 + \frac{1}{\frac{\mu^2+4\mu+7}{3}},$$

$$\frac{\mu^2+4\mu+7}{3} = 9 + \frac{\mu^2+4\mu-20}{3} = 9 + \frac{1}{\frac{10\mu^2+29\mu+24}{1}},$$

. . . . . .

Hence $29^3 - 6 \cdot 29 \cdot 10^2 - 7 \cdot 10^3 = -11$. From the chain

$$\frac{4-\mu}{11} = 1 - \frac{\mu+7}{11} = 1 - \frac{1}{\frac{\mu^2-7\mu+43}{28}},$$

$$\frac{\mu^2-7\mu+43}{28} = 2 - \frac{-\mu^2+7\mu+13}{28} = 2 - \frac{1}{\frac{2\mu^2-3\mu-7}{1}},$$

we have $(-3)^3 - 6(-3)2^2 - 7 \cdot 2^3 = -11$, a solution which belongs to the same root -7 (11). This time we compute the unit $16\mu^2 - 24\mu + 65$, and it is easy to see that we could not get the latter solution from $(\mu+7)/11$ in the regular chain, since for $\omega = 2\mu^2 - 3\mu - 8$ we have the equivalence $(\mu+7)/11 = (\omega+1)/(2\omega+1)$ but $\omega < 1$. With the congruence-root 2 we take immediately $2^3 - 6.2.1^2 - 7.1^3 = -11$, forming

$$\frac{\mu-2}{11} = 0 + \frac{\mu-2}{11} = 0 + \frac{1}{\frac{\mu^2+2\mu-2}{1}},$$

$$\frac{\mu^2+2\mu-2}{1} = 12 + \frac{\mu^2+2\mu-14}{1} = 12 + \frac{1}{\frac{12\mu^2+35\mu+26}{49}},$$

$$\frac{12\mu^2+35\mu+26}{49} = 4 + \frac{12\mu^2+35\mu-170}{49} = 4 + \frac{1}{\frac{49\mu^2+142\mu-14}{537}},$$

$$\frac{49\mu^2+149\mu-14}{537} = 1+\frac{49\mu^2+142\mu-551}{537} = 1+\frac{1}{\frac{61\mu^2+177\mu-193}{424}},$$

$$\frac{61\mu^2+177\mu-193}{424} = 1+\frac{61\mu^2+177\mu-617}{424} = 1+\frac{1}{\frac{10\mu^2+29\mu-43}{121}}.$$

. . . . . .

It seems to lead to a contradiction that the solution (10, 29), belonging to the root -7, has found a place in the chain $(\mu-2)/11$ above, but it can be explained as follows. From $(29\cdot 11)^3-6(29\cdot 11)(10\cdot 11)^2-7(10\cdot 11)^3=-11^4$ we deduce

$$(29-10\mu)\,[110(10\mu^2+29\mu-\gamma)/11^2+(29^2-6\cdot 10^2+10\gamma)/11] \;=\; -1.$$

Take $\omega = (10\mu^2+29\mu-\gamma)/11^2$ and $29-10(\mu-2)-20 = 9-10(\mu-2)$; then we have

$$10(\mu-2) \;=\; 9+\frac{1}{110\omega+x} = \frac{110\cdot 9\omega+9x+1}{110\omega+x}$$

and, having determined $x>0$ from $9x+1\equiv 0\ (110)$ as $x = 61$, we compute γ from $29^2-6\cdot 10^2+10\gamma = 61\cdot 11$, i. e., $\gamma = 43$; hence the improper equivalence above

$$\frac{\mu-2}{11} = \frac{9\omega+5}{110\omega+61},\quad \omega>1.$$

In a similar way we have

$$\frac{\mu+6}{11} = \frac{89\omega'+17}{110\omega'+21},\qquad \omega' = \frac{10\mu^2+29\mu+1}{11^2}>1,$$

but these improper equivalences are exceptional.* I have given these few examples to illustrate the use of these regular chains in the theory of cubic forms, and after this I am going to prove the statement above, that the solutions of the relation (1′) must be gotten by a regular formation of chain, except for some readily determined couple of values (α_i, β_i), $i\leqq m$ But I shall insert here incidentally the manner in which our relations are formed in the case of the general equation

$$A_0x^3+A_1x^2+A_2x+A_3 \;=\; 0,$$
$$A_0\beta^3+A_1\beta^2\alpha+A_2\beta\alpha^2+A_3\alpha^3 \;=\; \mp\, R.$$

* and cannot appear in a chain (12).

We have

$$(\beta-\alpha\mu)[\alpha(A_0\alpha\mu^2+A_0\beta\mu+A_1\alpha^2\mu+\gamma)+A_0\beta^2+A_1\alpha\beta+A_2\alpha^2-\alpha\gamma]=\mp R,$$

and, by $\beta = y_x R - z_x P_\varrho$, $\omega = A_0\alpha\mu^2+(A_0\beta+A_1\alpha^2)\mu+\gamma$, the already known equivalence

$$\frac{\mu+P_\varrho}{R_\varrho} = \frac{y_x\omega+y_{x-1}}{z_x\omega+z_{x-1}},$$

where

$$y_{x-1} = \left[\frac{\beta+\alpha P_\varrho}{R}(A_0\beta^2+A_1\alpha\beta+A_2\alpha^2-\alpha\gamma)\pm 1\right]\frac{1}{\alpha}.$$

Now we shall first show the existence of some cubic forms with no exceptional solutions. Let us consider the equation

$$f(x) = x^3+A_2x-A_3 = 0,$$

$A_2>0$, $A_3>0$. Since $27A_3^2+4A_2^3>0$, there exists only one real root $\mu>0$. From

$$\left(\frac{\beta}{\alpha}-\mu\right)\left(\frac{\beta^2}{\alpha^2}+\frac{\beta}{\alpha}\mu+\mu^2+A_2\right) = \frac{R}{\alpha^3},$$

and

$$\frac{\beta^2}{\alpha^2}+\frac{\beta}{\alpha}\mu+\mu^2+A_2 = \left(\frac{\beta}{\alpha}+\frac{\mu}{2}\right)^2+\frac{3\mu^2+4A_2}{4},$$

we have for all combinations of sign α, $\beta \gtrless 0$

$$\frac{\beta^2}{\alpha^2}+\frac{\beta}{\alpha}\mu+\mu^2+A_2>A_2\geqq 1,$$

whence

$$\left|\alpha\mu^2+\beta\mu+\frac{\beta^2+\alpha^2A_2}{\alpha}\right|>|\alpha|\geqq 2,$$

with the exception of $A_2=1$, $|\alpha|=1$, which may be examined separately. Hence by (16) $|\beta/\alpha-\mu|<R/2\alpha^2$ or, since $\beta = Ry_x-z_xP_\varrho$, $z_x=\alpha$,

$$\left|\frac{y_x}{z_x}-\frac{\mu+P_\varrho}{R_\varrho}\right|<\frac{1}{2\alpha^2};$$

and from this inequality we infer, as before, that y_x/z_x are fractions in the regular chain of $(\mu+P_\varrho)/R_\varrho$. Hence in this case all existing solutions are

attained by the usual formation of a regular chain. Let us now consider the equation

$$f(x) = x^3 - A_2 x - A_3 = 0,$$

$A_2 > 0$, $A_3 > 0$, $27A_3^2 - 4A_2^3 > 0$. The real root being $\mu > 0$ we compute for the complex one $u + iv$ from $\mu + u + iv + u - iv = 0$, $\mu(u^2 + v^2) = A_3$ the expressions $u = -\mu/2$ and $v = \frac{1}{2}\sqrt{3\mu^2 - 4A_2}$. Hence we must have $3\mu^2 - 4A_2 > 0$. For $3\mu^2 - 4A_2 > 4$, $\mu > 2\sqrt{(A_2+1)/3}$ we conclude, as above, that there cannot exist any exceptional solutions. But generally we have $f'(\delta_1) = \beta^2/\alpha^2 + \beta/\alpha\mu + \mu^2 - A_2 = (\beta/\alpha + \mu/2)^2 + \frac{1}{4}(3\mu^2 - 4A_2) > 0$ for all values $\alpha, \beta \gtrless 0$. Further, since $f(0) = -A_3 < 0$ and $f(1) = 1 - A_2 - A_3 \geqq 0$, $1 - A_2 \geqq A_3$ would give $A_2 = 0, 1$ and $A_3 = 1, 0$ with $f(x)$ reducible, we have $\mu > 1$, and if $\beta/\alpha > 0$ we are led to the inequality $\beta^2/\alpha^2 + \beta/\alpha\,\mu + \mu^2 - A_2 > \beta/\alpha$, whence $|\beta/\alpha - \mu| < R/\beta\alpha^2$ having for $\beta \geqq 2$ the same conclusion as before. After this preliminary examination we shall investigate the general case, and for that purpose we trace the two curves

$$f(x, y) = y^3 - A_2 y x^2 + A_3 x^3 = R > 0, \tag{17'}$$

$$yx + \mu x^2 = R/2. \tag{17''}$$

Obviously both curves have the asymptote $y + \mu x = 0$. Further, the values $x = 0$, $y = \sqrt[3]{R}$ from $\partial f/\partial x = 0 = x(-2A_2 y + 3A_3 x)$ afford a minimum, since $(d^2y/dx^2) = -\partial^2 f/\partial x^2 / \partial f/\partial y = -(-2A_2 y + 6A_3 x)/(3y^2 - A_2 x^2) = +2A_2/3\sqrt[3]{R} > 0$; $y = x\,3A_3/2A_2$ with $x = 2A_2\sqrt[3]{R/A_3(27A_3^2 - 4A_2^3)} > 0$ gives rise to a maximum. In a similar way we have from $\partial f/\partial y = 0 = 3y^2 - A_2 x^2$, $y = x\sqrt{A_2/3}$, $x = \sqrt[3]{R\Big/\left(A_3 - \frac{2A_2}{3}\sqrt{\frac{A_2}{3}}\right)} > 0$ giving $(d^2x/dy^2) = -6y/x(-2A_2 y + 3A_3 x) < 0$, hence a maximum; and $y = -x\sqrt{A_2/3}$, $x = \sqrt[3]{R\Big/\left(A_3 + \frac{2A_2}{3}\sqrt{\frac{A_2}{3}}\right)}$, yielding a minimum. Thus the curve has the form shown in Fig. 1. We see that the points of intersection between (17') and (17'') are given by

$$(y + \mu x)(y^2 - \mu x y + \mu^2 x^2 - A_2 x^2) = 2x(y + \mu x),$$

or

$$(y - \mu x/2)^2 + 3/4\,\mu^2 x^2 - A_2 x^2 = 2x$$

that is

$$(R - 3\mu x^2)^2 - 8x^3 + 4x^4(3/4\,\mu^2 - A_2) = 0$$

This equation must have no or two real roots, if there exists any intersection near the y axis; hence the other real root falls at the asymptote $y+\mu x = 0$. We can also control this fact in the following manner. The straight line $y-mx = 0$ gives with the curve (17″) the point $x_2 = \sqrt[2]{R/2(m+\mu)}$ and with (17′) $x_3 = \sqrt[3]{R/(m^3-A_2 m+A_3)}$. If the two curves possess any intersection at the asymptote, there must exist ordinates $x_2 \geqq x_3$, that is

$$R^2/(m^3-A_2 m+A_3)^2 \leqq R^3/8(m+\mu)^3$$

or

$$R \geqq 8(m+\mu)/(m^2-\mu m+\mu^2-A_2)^2, \tag{18}$$

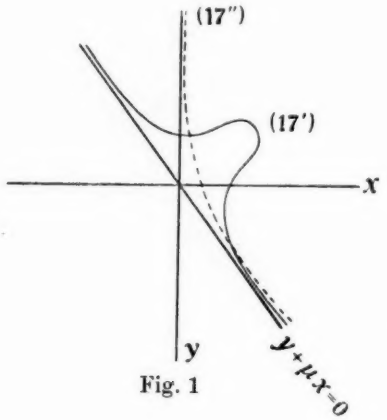

Fig. 1

which inequality is satisfied, m converging to $-\mu$. Finally we might also have seen this fact by observing (and this is important for the investigations of higher forms than cubic) that the area between the asymptote and curve (17″) is infinite while that between the asymptote and curve (17′) is finite. Hence, for example, all $x > 0$, $y < 0$ for which $y/x+\mu < R/2x^2$ lie on the branch (17′) between the asymptote and (17″), and just these values must be attained by formation of the regular chain; and the eventual exceptional values fall on that part of (17′) which is traced as a loop within (17″). Hence we have in this case to seek exceptional solutions among the integers

$$0<\beta \leqq 3A_3\sqrt[3]{R/A_3(27A_3^2-4A_2^3)}, \quad 0<\alpha \leqq \sqrt[3]{R/(A_3-2A_2/3\sqrt{A_2/3})}, \tag{19}$$

but putting, for example, $m = \sqrt{A_2/3}$ instead of the real maximum of (18) we obtain an upper limit for R, excluding exceptional values, namely

(19′) $$R > 8(\mu + \sqrt{A_2/3}) / (\mu^2 - \mu\sqrt{A_2/3} - 2A_2/3)^2,$$

which is relatively good.

Example. $x^3 - 6x + 7 = 0, \quad R = 11.$

By means of (19) we have to try $\beta \leqq 3\sqrt[3]{539/459}$, that is, $\beta = 1, 2, 3$ and $\alpha \leqq \sqrt{11/1,4}$, that is, $\alpha = 1, 2$, and actually (2, 3) gives an exceptional solution, as we have seen above. It only remains to discuss similarly the case of three real roots. Thus we start from the equation

$$f(x) = x^3 - A_2 x + A_3 = 0,$$

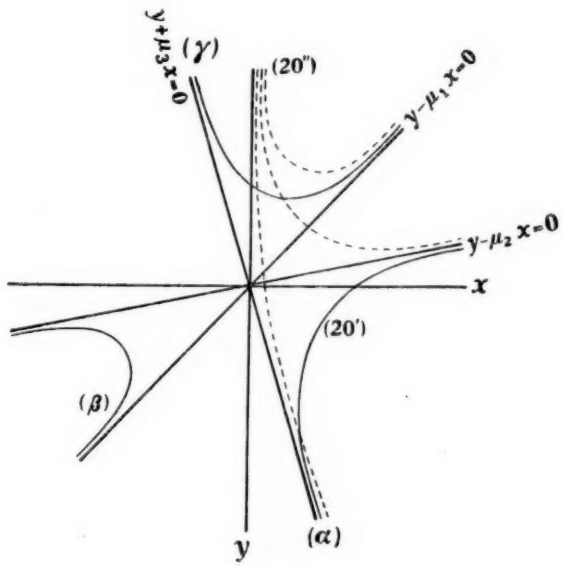

Fig. 2

where $27A_3^2 - 4A_2^3 < 0$, and trace the both curves

(20′) $$y^3 - A_2 y x^2 + A_3 x^3 = R$$

(20″) $$yx - \mu_i x^2 = R/2.$$

The curve (20′) has three real asymptotes $y - \mu_1 x = 0$, $y - \mu_2 x = 0$, $y + \mu_3 x = 0$. Relatively to the axis of x we have a minimum at the point $(0, \sqrt[3]{R})$ and a maximum at $\left\{2A_2\sqrt[3]{R/(27A_3^2 - 4A_2^3)A_3}, 3A_3\sqrt[3]{R/A_3(27A_3^2 - 4A_2^3)}\right\}$, placed in the third quadrant. Relatively to the axis of y we compute the minimum

$$\left\{\sqrt[3]{R/(A_3+A_2/3\sqrt{2A_2/3})},\quad -\sqrt{2A_2/3}\sqrt[3]{R/(A_3+A_2/3\sqrt{2A_2/3})}\right\}$$

and a maximum in

$$\left\{\sqrt[3]{R/(A_3-A_2/3\sqrt{2A_2/3})},\quad \sqrt{2A_2/3}\sqrt[3]{R/(A_3-A_2/3\sqrt{2A_2/3})}\right\},$$

hence placed in the fourth and third quadrant respectively. Hence the curve (20′) will be as in the Fig. 2. This curve must have three branches, and it is interesting to compare it with (17′). We see how the two asymptotes disappear with the two real roots, and how the branch (β) becomes a loop connected to (α) and (γ). In the figure the hyperbolas are traced as dotted lines, and for $R<0$ we obtain the symmetrical curve. Now it is fundamental that there always meet at each asymptote one branch of (20′) and one of the type (20″), and that the branch of (20′) must always come nearer the asymptote, which we prove in the same way as above. Hence we conclude, that all exceptional values must fall below certain limits α_i and β_i which are fully determined for each form and each R. Further we see that, for instance, no solutions (β_i, α_i) on the branch (α) can be reached by proceeding only from the root μ_1, but that we obtain all solutions proceeding from all three roots, with exceptions in the case of some few values, by means of the usual, regular formation of chain; and it was just this result we had in view.

Example.

$$x^3-3x+1=0,\quad \mu_1\sim 1.53,\quad \mu_2\sim 0.347,\quad -\mu_3\sim 1.9,\quad R=1.$$

$$\mu_1=1+\frac{\mu_1-1}{1}=1+\cfrac{1}{\cfrac{\mu_1^2+\mu_1-2}{1}},$$

$$\frac{\mu_1^2+\mu_1-2}{1}=1+\frac{\mu_1^2+\mu_1-3}{1}=1+\cfrac{1}{\cfrac{\mu_1^2+2\mu_1-2}{3}},$$

$$\frac{\mu_1^2+2\mu_1-2}{3}=1+\frac{\mu_1^2-2\mu_1-5}{3}=1+\cfrac{1}{\cfrac{2\mu_1^2+3\mu_1-2}{1}},$$

$$\frac{2\mu_1^2+3\mu_1-2}{1}=7+\frac{2\mu_1^2+3\mu_1-9}{1}=7+\cfrac{1}{\cfrac{15\mu_1^2+23\mu_1-12}{17}},$$

$$\frac{15\mu_1^2+23\mu_1-12}{17}=3+\frac{15\mu_1^2+23\mu_1-63}{17}=3+\cfrac{1}{\cfrac{47\mu_1^2+72\mu_1-54}{73}},$$

$$\frac{47\mu_1^2+72\mu_1-54}{73} = 2+\frac{47\mu_1^2+72\mu_1-200}{73} = 2+\frac{1}{\frac{109\mu_1^2+167\mu_1-119}{111}},$$

$$\frac{109\mu_1^2+167\mu_1-119}{111} = 3+\frac{109\mu_1^2+167\mu_1-452}{111} = 3+\frac{1}{\frac{374\mu_1^2+573\mu_1-449}{703}},$$

$$\frac{374\mu_1^2+573\mu_1-449}{703} = 1+\frac{374\mu_1^2+573\mu_1-1152}{703} = 1+\frac{1}{\frac{483\mu_1^2+740\mu_1-1095}{1007}},$$

$$\frac{483\mu_1^2+740\mu_1-1095}{1007} = 1+\frac{483\mu_1^2+740\mu_1-2102}{1007} = 1+\frac{1}{\frac{857\mu_1^2+1313\mu_1-853}{521}},$$

$$\frac{857\mu_1^2+1313\mu_1-853}{521} = 6+\frac{857\mu_1^2+1313\mu_1-3979}{521} = 6+\frac{1}{\frac{5625\mu_1^2+8618\mu_1-3962}{1907}},$$

. . . . . .

from which we take the two solutions $1^3-1\cdot1^2\cdot3+1\cdot1^3=-1$ and $3^3-3\cdot3\cdot2^2+2^3=-1$. The latter solution gives rise to the equivalence $\mu_1=(3\omega+2)/(2\omega+1)$, $\omega=2\mu_1^2+3\mu_1-2>1$. By means of the root μ_2 the solution will give the same equivalence and $\omega_1=2\mu_2^2+3\mu_2-2$, which is negative and absolutely less than one, whence no regular formation of a chain from μ_2 could have led us to this solution. Incidentally we observe that the fraction $8618/5625 = 1+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|7}+\frac{1|}{|3}+\frac{1|}{|2}+\frac{1|}{|3}+\frac{1|}{|1}+\frac{1|}{|1}+\frac{1|}{|6}$ will give a good approximation to μ_1.* From μ_2 we have the chain

$$\mu_2 = 0+\mu_2 = 0+\frac{1}{\frac{-\mu_2^2+3}{1}},$$

$$\frac{-\mu_2^2+3}{1} = 2+\frac{-\mu_2^2+1}{1} = 2+\frac{1}{\frac{-2\mu_2^2-\mu_2+4}{3}},$$

* See formula (12'), p. 255.

$$\frac{-2\mu_2^2-\mu_2+4}{3}=1+\frac{-2\mu_2^2-\mu_2+1}{3}=1+\frac{1}{\frac{-3\mu_2^2-\mu_2+8}{1}},$$

$$\frac{-3\mu_2^2-\mu_2+8}{1}=7+\frac{-3\mu_2^2-\mu_2+1}{1}=7+\frac{1}{\frac{-23\mu_2^2-8\mu_2+64}{17}},$$

$$\frac{-23\mu_2^2-8\mu_2+64}{17}=3+\frac{-23\mu_2^2-8\mu_2+13}{17}=3+\frac{1}{\frac{-72\mu_2^2-25\mu_2+184}{73}},$$

$$\frac{-72\mu_2^2-25\mu_2+184}{73}=2+\frac{-72\mu_2^2-25\mu_2+38}{73}=2+\frac{1}{\frac{-167\mu_2^2-58\mu_2+433}{111}},$$

. . . . . .

It is in fact remarkable that in this case where $\mu_1=-1/(\mu_2-1)$, that is $\mu_2=\mu_1^2-2$, in the two chains from μ_1 and μ_2 all $b_\nu(\nu\geqq 2)$ are equal, since generally from $\beta_i^3-3\beta_i\alpha_i^2+\alpha_i^3=R$ and $\beta_i=\alpha_i+\gamma_i$ there follows $\gamma_i^3-3\gamma_i\beta_i^2+\beta_i^3=-R$, and since, further, $\beta_i/\alpha_i=1+\gamma_i/\alpha_i$; $\alpha_i/\gamma_i=1+(\alpha_i-\gamma_i)/\gamma_i$; $\gamma_i/\beta_i=0+\gamma_i/(\alpha_i+\gamma_i)$; $(\alpha_i+\gamma_i)/\gamma_i=2+(\alpha_i-\gamma_i)/\gamma_i$. For $\mu_3\sim -1{,}9$ we have

$$\frac{2+\mu_3}{1}=0+\frac{2+\mu_3}{1}=0+\frac{1}{\frac{\mu_3^2-2\mu_3+1}{1}},$$

$$\frac{\mu_3^2-2\mu_3+1}{1}=8+\frac{\mu_3^2-2\mu_3-7}{1}=8+\frac{1}{\frac{8\mu_3^2-15\mu_3+2}{17}},$$

$$\frac{8\mu_3^2-15\mu_3+2}{17}=3+\frac{8\mu_3^2-15\mu_3-49}{17}=3+\frac{1}{\frac{25\mu_3^2-47\mu_3-10}{73}},$$

. . . . . .

Now $\mu_1=(\mu_3-1)/\mu_3$, that is $-\mu_3=\mu_1^2+\mu_1-2$, and also this formation is identical with the former for μ_1, which we see as follows. By means of $\beta_i^3-3\beta_i\alpha_i^2+\alpha_i^3=R$ and $\beta_i=\alpha_i+\gamma_i$ we compute $\alpha_i^3-3\alpha_i\gamma_i^2-\gamma_i^3=-R$ and further $\beta_i/\alpha_i=1+\gamma_i/\alpha_i$; $\alpha_i/\gamma_i=1+(\alpha_i-\gamma_i)/\gamma_i$; $\gamma_i/(\alpha_i-\gamma_i)=1+(2\gamma_i-\alpha_i)/(\alpha_i-\gamma_i)$; $(\alpha_i-\gamma_i)/(2\gamma_i-\alpha_i)=7+(8\alpha_i-15\gamma_i)/(2\gamma_i-\alpha_i)$; and $-\alpha_i/\gamma_i=-2+(2\gamma_i-\alpha_i)/\gamma_i$; $\gamma_i/(2\gamma_i-\alpha_i)=8+(8\alpha_i-15\gamma_i)/(2\gamma_i-\alpha_i)$.

We have found that the substitution $y=-y_1+x_1$, $x=-y_1$ transforms the form $y^3-3yx^2+x^3$ into itself and that the roots μ_1, μ_2 and μ_3 are

also expressed as $\mu_1 = (23\nu+3)/(15\nu+2)$, $\mu_2 = (8\nu+1)/(23\nu+3)$, $-\mu_3 = (15\nu+2)/(8\nu+1)$, where ν is the common continued fraction above. Then μ_1 must define a Galois corpus, which is obvious from the equation $x^3-3x+1=0$ itself, since $D = \Delta(\mu_1) = 9^2$ is a square. In this connection we ask: Which values of the coefficients A_2, A_3 in $x^3+A_2x+A_3=0$ give similar properties for the formation of chains, or also: Which cubic forms $y^3+A_2yx^2+A_3x^3$ have any transformation into themselves? The answer is: The equation $x^3+A_2x+A_3=0$ must define a Galois corpus; hence the relation $-(27A_3^2+4A_2^3) = \Delta^2$, whence $\mu_1 = \frac{1}{2}\left[-\mu_2 \pm \frac{\Delta}{f'(\mu_2)}\right]$. Then we have to discuss the conditions for the identity

$$\mu_1 = \frac{1}{2}\left[-\mu_2 + \frac{\Delta}{3\mu_2^2+A_2}\right] = \frac{1}{2}\,\frac{2A_2\mu_2+(3A_3+\Delta)}{3\mu_2^2+A_2} = \frac{\alpha\mu_2+\beta}{\gamma\mu_2+\delta}.$$

A simple investigation gives the result:

1. $A_2 = mn$, $3A_3 = ns$, $\Delta = n$, $-4m^3n^3-3n^2s^2 = n^2$, hence

$$n = -\frac{3s^2+1}{4m^3} \text{ and } \alpha = -\frac{3s+1}{2},\ \beta = -\frac{3s^2+1}{4m},\ \gamma = 3m,\ \delta = \frac{3s-1}{2}.$$

2. $A_2 = mn$, $3A_3 = ns$, $\Delta = 3n$, hence for $n = 3n_1$ we compute

$$n_1 = -\frac{s^3+3}{4m^3},\ \alpha = -\frac{s+1}{2},\ \beta = -\frac{s^2+3}{4m},\ \gamma = m,\ \delta = \frac{s-1}{2}.$$

Example. $s=-3$, $m=1$, $A_2=-7$, $A_3=7$ and $\alpha=4$, $\delta=-5$, $\beta=-7$, $\gamma=3$, hence $\mu_\tau = (4\mu_i-7)/(3\mu_i-5)$ and the substitution $x = 4\xi-7\eta$, $y = 3\xi-5\eta$ transforms $x^3-7xy^2+7y^3$ in $\xi^3-7\xi\eta^2+7\eta^3$. For $s=+3$ we have $\alpha=-5$, $\beta=-7$, $\gamma=3$, $\delta=4$, $\mu_\tau = (-5\mu_i-7)/(3\mu_i+4)$, and we readily verify that $x=-5\xi-7\eta$, $y=3\xi+4\eta$ transforms the cubic form $x^3-7xy^2-7y^3$ into $\xi^3-7\xi\eta^2-7\eta^3$, and that $4^3-7\cdot4\cdot3^2+7\cdot3^3=1$, $(-5)^3+7\cdot5\cdot3^2-7\cdot3^3=+1$. For $s=-1$ we find in the case (2) just the equation, $x^3-3x+1=0$, treated above.

From a somewhat different point of view we may reason as follows: A necessary condition for an equivalence

$$\mu_i = \frac{\alpha\mu_\tau+\beta}{\gamma\mu_\tau+\delta}, \qquad \alpha\delta-\beta\gamma = \pm 1, \tag{21}$$

where μ_i and μ_τ satisfy $f_3(x) = x^3 + A_2 x + A_3 = 0$, is that $K(\mu_i)$ will form a Galois corpus. Hence there must exist a relation $\sigma\mu_\tau = \varrho\mu_i^2 + \chi\mu_i + \tau$, where σ, ϱ, χ and τ are integers. Comparing with Theorem 1 this μ_τ will in the equivalence (21) give rise to the equation $\chi^3 + A_2\chi\varrho^2 + A_3\varrho^3 = \pm\sigma$. And conversely, from that relation and $\sigma\mu_\tau = \varrho\mu_i^2 + \chi\mu_i + \tau$ we form the equivalence (21). Hence, the necessary and sufficient condition that a form $y^3 + A_2 y x^2 + A_3 x^3$ may have a transformation in itself, is that $K(\mu_i)$ has the property of a Galois corpus, whence a relation $\sigma\mu_\tau = \alpha\mu_i^2 + \beta\mu_i + \gamma$, where σ, α, β and γ are integers and, further that σ, α and β are connected in the equation $\beta^3 + A_2\beta\alpha^2 + A_3\alpha^3 = \pm\sigma$.

An analogous theorem is true even for any form $f_m(y, x) = y^m + + A_2 y^{m-2} x^2 + \cdots + A_m x^m$ of higher degree than the third.

From this investigation it is readily seen how the results above may be generalized in the case of forms $f_m(x, y) = A_0 y^m + A_1 y^{m-1} x + \cdots + A_m x^m$ of any higher degree m than the third, and how the integer solutions of $f_m(x, y) = R$ for given R are found in a similar way when $f_m(x) = 0$ has at least one real root. If the irreducible equation $f_m(x) = 0$ has no real root, every branch of the curve $f_m(x, y) = R$ must be traced in the finite part of the plane, whence the eventual integer solutions of $f_m(x, y) = R$ are to be sought among values $x, y \leqq N$, where N can be readily determined.

MALMÖ, SWEDEN,
October 1924.

A PROPERTY OF SEQUENCES OF LAPLACE.

By Horace L. Olson.

Wilczynski has shown* that a congruence of straight lines can be represented, from the projective point of view, by a system of partial differential equations

$$(D) \qquad \begin{aligned} y_v &= mz, \quad z_u = ny, \quad y_{uu} = ay + bz + cy_u + dz_v, \\ z_{vv} &= a'y + b'z + c'y_u + d'z_v, \end{aligned}$$

in which the coefficients are functions of u and v, and subscripts denote differentiation. This system will have precisely four linearly independent pairs of solutions $(y^{(k)}, z^{(k)})$ $(k = 1 \cdots 4)$, provided that the coefficients satisfy the integrability conditions

$$(I) \qquad \begin{aligned} &c = f_u, \quad d' = f_v, \quad b = -d_v - df_v, \quad a' = -c'_u - c'f_u, \quad mn - c'd = f_{uv}, \\ &m_{uu} + d_{vv} + df_{vv} + d_v f_v - f_u m_u = ma + db', \\ &n_{vv} + c'_{uu} + c'f_{uu} + c'_u f_u - f_v n_v = c'a + nb', \\ &2m_u n + mn_u = a_v + f_u mn + a'd, \quad m_v n + 2mn_v = b'_u + f_v mn + bc', \end{aligned}$$

where f is a function of u and v.

The most general pair of solutions of system (D) will then be expressible in the form

$$y = \sum_{k=1}^{4} c^{(k)} y^{(k)}, \qquad z = \sum_{k=1}^{4} c^{(k)} z^{(k)}.\dagger$$

If $y^{(1)} \cdots y^{(4)}$, $z^{(1)} \cdots z^{(4)}$ be interpreted as the homogeneous coördinates of two points P_y and P_z, then, as u and v vary, these points will describe two surfaces, S_y and S_z (either or both of which may be degenerate), and the line $P_y P_z$ will generate a congruence whose focal surface consists of S_y and S_z. The ruled surfaces obtained by equating u or v to a constant are its developables. Any congruence whose focal surface consists of two distinct sheets can be studied from the projective point of view by this method.

Any transformation of the form

$$y = \lambda(u)\overline{y}, \quad z = \mu(v)\overline{z}, \quad \overline{u} = \varphi(u), \quad \overline{v} = \psi(v),$$

* E. J. Wilczynski, Sur la théorie générale des congruences. Mémoire couronné par la Classe des Sciences, Acad. Roy. de Belg., 2e ser., t. 3 (1911) p. 16.

† Ibid. p. 17.

where λ, μ, φ, and ψ are functions of the variables indicated, will convert system (D) into another system of the same form. A function of the coefficients and variables and their derivatives which is left unchanged (except possibly for multiplication by a factor) under all such transformations is called a covariant or an invariant according as it does or does not involve the dependent variables. The projective properties of the congruence are determined by the invariants and covariants of its system of equations (D).*

It is the purpose of this paper to study the geometry of two covariants which, to the best of the author's knowledge, have not yet been discussed. As a preliminary step let us consider the first Laplace transform of the parametric net on S_y,

$$\varrho = y_u - \frac{m_u}{m} y, \tag{1}$$

which represents a point which traverses the second sheet, S_ϱ, of the focal surface of the congruence consisting of the tangents to the curves $v =$ constant on S_y.† Similarly the minus first Laplace transform of the parametric net on S_z,

$$\sigma = z_v - \frac{n_v}{n} z, \tag{2}$$

represents a point which traverses the second sheet S_σ of the focal surface of the congruence consisting of the tangents to the curves $u =$ constant on S_z.‡ The tangent plane to S_ϱ at any one of its points intersects the line $P_z P_\sigma$ corresponding to it in the minus first Laplace transform in a point P_ξ. The tangent plane to S_σ at any one of its points intersects the corresponding line $P_y P_\varrho$ of the first Laplace transform in a point P_η.

A general point on the tangent plane to S_ϱ at P_ϱ is given by

$$\varrho + p\varrho_u + q\varrho_v = \left\{ -\frac{m_u}{m} + p\left(a + \frac{m_u^2}{m^2} - \frac{m_{uu}}{m}\right) + q\left(\frac{m_u m_v}{m^2} - \frac{m_{uv}}{m} + mn\right)\right\} \times y + pbz + \left\{1 + q\left(c - \frac{m_u}{m}\right)\right\} y_u + dz_v,$$

where p and q are independent variables. To obtain the point in which this plane intersects the line $P_z P_\sigma$ we must choose p and q so that the coefficients of y and y_u shall vanish, viz.,

$$p = \frac{m}{m_u - cm}, \quad q = \frac{m^2(m_{uu} - cm_u - am)}{(m_u - cm)\,(m_u m_v - m m_{uv} + m^3 n)}.$$

* Ibid. §§ 1-3.

† Ibid. p. 23.

‡ Ibid. p. 23.

Since our coördinates are homogeneous, we can represent P_ξ by

$$\xi = z_v + \frac{b}{d} z = z_v - \left(\frac{d_v}{d} + f_v\right) z. \tag{3}$$

Similarly, or by considerations of symmetry, we find

$$\eta = y_u + \frac{a'}{c'} y = y_u - \left(\frac{c'_u}{c'} + f_u\right) y. \tag{4}$$

It can easily be verified by means of the transformation formulae* that the functions η and ξ are covariants of system (D).

If m or n or both vanish identically (i. e., if one or both sheets of the focal locus are curves), at least one of the Laplace transforms of the given congruence ceases to exist and the geometrical definition of the line $P_\eta P_\xi$ which we have given breaks down, although the expressions (3) and (4) for η and ξ remain valid. This fact suggests that there is a geometrical definition of the points P_η and P_ξ which is valid in these special cases. By the use of the third and fourth of the differential equations (D) we can express η and ξ in the forms

$$\eta = \frac{1}{c'}(z_{vv} - b'z - d'z_v), \qquad \xi = \frac{1}{d}(y_{uu} - ay - cy_u). \tag{5}$$

Hence, if neither c' nor d vanishes identically (whether m or n vanishes or not), P_η is the point of intersection of the tangent line to a curve $v =$ constant on S_y at one of its points P_y with the osculating plane to the corresponding curve $u =$ constant on S_z at the point P_z corresponding to P_y. Similarly P_ξ is the point of intersection of the tangent line to a curve $u =$ constant on S_z at one of its points P_z with the osculating plane to the corresponding curve $v =$ constant on S_y at the corresponding point P_y. If S_y degenerates to a curve C_y, then a curve $v =$ constant on S_y must be understood to mean C_y itself; similarly, if S_z degenerates to a curve C_z, then a curve $u =$ constant on S_z must be understood to mean C_z itself. If c' or d or both vanish identically, (i. e., if one or both sheets of the focal locus are developables), both the analysis and the geometry of this paper break down, since one or both of the surfaces S_ϱ, S_σ become curves.

* Ibid. § 16, p. 20.

From equations (3), (4), and (D) we find

$$\eta_u = A_1 y - \frac{c'_u}{c'}\eta + d\xi, \quad \eta_v = A_2 y + B_2 z,$$

(6)

$$\xi_u = A_3 y + B_3 z, \qquad \xi_v = B_4 z + c'\eta - \frac{d_v}{d}\xi,$$

where

$$A_1 = a - \frac{c'_{uu}}{c'} - f_{uu} - \frac{c'_u}{c'} f_u = \frac{1}{c'}\{n_{vv} - f_v n_v - nb'\},$$

$$A_2 = c'd - \frac{\partial^2}{\partial_u \partial_v}\log c', \qquad A_3 = n_v - n\frac{d_v}{d} - nf_v,$$

(7)

$$B_2 = m_u - m\frac{c'_u}{c'} - mf_u, \qquad B_3 = c'd - \frac{\partial^2}{\partial_u \partial_v}\log d,$$

$$B_4 = b' - \frac{d_{vv}}{d} - f_{vv} - \frac{d_v}{d} f_v = \frac{1}{d}\{m_{uu} - f_u m_u - ma\}.$$

From equation (6) follow at once:

THEOREM 1. *The tangent to a curve $v =$ constant at any point P_η on S_η intersects the corresponding line $P_y P_\xi$. The tangent to a curve $u =$ constant at any point P_ξ on S_ξ intersects the corresponding line $P_z P_\eta$.*

THEOREM 2. *The tangent to a curve $u =$ constant at any point P_η on S_η intersects the corresponding line $P_y P_z$. The tangent to a curve $v =$ constant at any point P_ξ on S_ξ intersects the corresponding line $P_y P_z$. These points of intersection coincide if and only if $A_2 B_3 - A_3 B_2 = 0$.*

Let us denote by $\bar{\Gamma}$ the congruence of lines $P_\eta P_\xi$. In order to determine the focal locus of this congruence we make a transformation

$$\eta = \alpha\bar{\eta} + \beta\bar{\xi}, \quad \xi = \gamma\bar{\eta} + \delta\bar{\xi}, \tag{8}$$

$\bar{\eta}$ and $\bar{\xi}$ will then represent two points $P_{\bar{\eta}}$ and $P_{\bar{\xi}}$, on each line of $\bar{\Gamma}$.

If α, β, γ, and δ are so chosen that there are linear homogeneous relations among $\bar{\eta}$, $\bar{\eta}_u$, $\bar{\eta}_v$, $\bar{\xi}$ and $\bar{\xi}$, $\bar{\xi}_u$, $\bar{\xi}_v$, $\bar{\eta}$ respectively, the lines of $\bar{\Gamma}$ will be common tangents of the surfaces $S_{\bar{\eta}}$ and $S_{\bar{\xi}}$ generated by the points $P_{\bar{\eta}}$ and $P_{\bar{\xi}}$, or, in other words, these surfaces will be the two sheets of the focal surface of $\bar{\Gamma}$. If we apply this transformation to equations (6) and eliminate y and z from the first, second, and fourth and from the first, third, and fourth of the resulting equations, we obtain

$$A_2 B_4\Big(\alpha\bar{\eta}_u + \beta\bar{\xi}\bar{\xi}_u + \alpha_u\bar{\eta} + \beta_u\bar{\xi} + \frac{c'_u}{c'}\alpha\bar{\eta} + \frac{c'_u}{c'}\beta\bar{\xi} - d\gamma\bar{\eta} - d\delta\bar{\xi}\Big) + A_1 B_2\Big(\gamma\bar{\eta}_v + \delta\bar{\xi}\bar{\xi}_v$$

(9a)

$$+\gamma_v\bar{\eta} + \delta_v\bar{\xi} - c'\alpha\bar{\eta} - c'\beta\bar{\xi} + \frac{d_v}{d}\gamma\bar{\eta} + \frac{d_v}{d}\delta\bar{\xi}\Big) - A_1 B_4(\alpha\bar{\eta}_v + \beta\bar{\xi}\bar{\xi} + \alpha_v\bar{\eta} + \beta_v\bar{\xi}) = 0,$$

$$A_3B_4\left(\alpha\bar{\eta}_u+\beta\bar{\xi}_u+\alpha_u\bar{\eta}+\beta_u\bar{\xi}+\frac{c'_u}{c'}\alpha\bar{\eta}+\frac{c'_u}{c'}\beta\bar{\xi}-d\gamma\bar{\eta}-d\delta\bar{\xi}\right)+A_1B_3\left(\gamma\bar{\eta}_v+\delta\bar{\xi}_v\right.$$
(9b)
$$\left.+\gamma_v\bar{\eta}+\delta_v\bar{\xi}-c'\alpha\bar{\eta}-c'\beta\bar{\xi}+\frac{d_v}{d}\gamma\bar{\eta}+\frac{d_v}{d}\delta\bar{\xi}\right)-A_1B_4(\gamma\bar{\eta}_u+\delta\bar{\xi}_u+\gamma_u\bar{\eta}+\delta_u\bar{\xi})=0.$$

By the use of equations (8) and (9) we deduce

THEOREM 3. *The foci of any line $P_\eta P_\xi$ are given by the factors of the covariant*

$$A_3B_4\eta^2+(A_2B_3-A_3B_2-A_1B_4)\eta\xi+A_1B_2\xi^2. \tag{10}$$

To find the curves in which the developable surfaces of $\bar{\Gamma}$ cut the surfaces S_y and S_z, we make a transformation

$$\bar{u}=\varphi(u,v), \qquad \bar{v}=\psi(u,v). \tag{11}$$

If φ and ψ are so chosen that the sets of functions η, ξ, $\eta_{\bar{v}}$, $\xi_{\bar{v}}$ and η, ξ, $\eta_{\bar{u}}$, $\xi_{\bar{u}}$ are connected by linear homogeneous relations, then $u=$ constant and $v=$ constant will determine the developables of $\bar{\Gamma}$. By means of equations (6) and (11) we deduce

THEOREM 4. *The developables of $\bar{\Gamma}$ are determined by $\varphi=$ constant and $\psi=$ constant, where φ and ψ are independent solutions of the differential equation**

$$A_2B_4\omega_u^2-(A_2B_3-A_3B_2+A_1B_4)\omega_u\omega_v+A_1B_3\omega_v^2=0. \tag{12}$$

* The congruence $\bar{\Gamma}$ is the "joint-axis" congruence discussed briefly by Wilczynski in his paper, The general theory of congruences, Trans. Amer. Math. Soc., vol. 16 (1915), p. 326.

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GREEN'S LEMMA.

By H. E. Bray.

1. **Introduction.** It has been pointed out by E. B. Van Vleck* that many of the current proofs of Green's Lemma, viz.

$$\int\int_S \frac{\partial Q}{\partial x}\,dx\,dy = \int_C Q\,dy,$$

introduce undesirable restrictions on the nature of the contour C of the region S. The object of this article is to remove some of these restrictions, and also some of the restrictions on the function Q, at the same time making the proof more systematic by basing it upon well established theorems regarding the Lebesgue and Stieltjes integral and the general properties of continuous curves. It will be seen that the lemma, thus generalized, follows readily from these fundamental theorems. In the last section the formula is discussed in connection with the Stieltjes Potential function,

$$\Phi = \int_E \log\frac{1}{r}\,df(e).$$

The contour C which we shall consider is a simple closed curve interior to the fundamental region $R[0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1]$. It is defined by the equations

$$x = \varphi(t),\ y = \psi(t), \qquad a \leqq t \leqq a + T = b,$$

where φ, ψ are continuous functions of period T, i. e.,

$$\varphi(t+T) = \varphi(t), \qquad \psi(t+T) = \psi(t).$$

C being simple we assume that there is no smaller positive number than T for which the last system of equations is satisfied. In order that C may be quadrable and that the Stieltjes integral in the second member may have a meaning, we assume that $\psi(t)$ is a function of limited variation in its interval of periodicity. Later we assume that C is rectifiable, in which case, of course, both φ and ψ are functions of limited variation.

We make use of the following properties of C which have been established by de la Vallée Poussin.

* E. B. Van Vleck, These Annals, vol. 22 (1921), pp. 226-237.

Property 1.* Since C is quadrable, it is possible to draw a polygon P' entirely within C and a polygon P'' enclosing C within its interior such that the difference of their areas, $P''-P'$, will be arbitrarily small and neither P' nor P'' will have a point in common with C.

Property 2.† Since C is a simple closed curve, it is possible to inscribe a polygon P, with vertices

$$[\varphi(t_i),\ \psi(t_i)];\quad i=1, 2, \cdots n;\quad t_0<t_1<t_2\cdots<t_n=t_0+T,$$

such that:

(a) the vertices of P are encountered in the same order on P as on C,

(b) the differences t_i-t_{i-1} will be arbitrarily small,

(c) P will have no multiple point.

2. THEOREM 1. *If* $\frac{\partial Q}{dx}$ *is summable superficially, if* Q *is an absolutely continuous function of* x *for every* y, *if* Q *is a continuous function of* (x, y), *and if* C *is a simple closed curve defined by* $x=\varphi(t)$, $y=\psi(t)$, *with* ψ *a function of limited variation, then*

$$\int\int_S \frac{\partial Q}{\partial x}\,dx\,dy=\int_C Q\,dx, \tag{1}$$

where the first member is a Lebesgue and the second a Stieltjes integral, and where S *is the simply connected region enclosed by* C.

Since φ and ψ are continuous it is possible, according to Property 2, to inscribe a simple polygon P in C which entirely encloses P' and is entirely enclosed by P'', and such that, in view of the absolute continuity of the Lebesgue integral, the quantity

$$\left|\int\int_S \frac{\partial Q}{\partial x}\,dx\,dy-\int\int_P \frac{\partial Q}{\partial x}\,dx\,dy\right| \leqq 2\int\int_{P''-P'}\left|\frac{\partial Q}{\partial x}\right| dx\,dy \tag{2}$$

may be made as small as we please by a judicious choice of the values $t_0\cdots t_n$, the differences t_i-t_{i-1} being sufficiently small.

Let us consider an infinite sequence of inscribed polygons $\{P^j\}$ of the type just described, representing the vertices of P^j by means of the parameter values $t_0^j<t_1^j<t_2^j\cdots<t_{n_j}^j$, where

$$t_{n_j}^j=t_0^j+T,\ |t_i^j-t_{i-1}^j|\leqq\delta_j \quad\text{and}\quad \lim_{j=\infty}\delta_j=0.$$

* de la Vallee Poussin, Cours d'analyse, vol. 1, 3rd edit., pp. 384, 385.

† Ibid., p. 380; Remarque.

We can now state that, by (2),

$$\lim_{j=\infty} \int\int_{P^j} \frac{\partial Q}{\partial x} dx\, dy = \int\int_S \frac{\partial Q}{\partial x} dx\, dy.$$

Next consider the Stieltjes integral in (1). Let C^j represent the contour of P^j. C^j is a regular curve to which Green's Lemma is readily applicable.* Consequently

$$\int\int_{P^j} \frac{\partial Q}{\partial x} dx\, dy = \int\int_{C^j} Q(x,y)\, dy.$$

Here we suppose that the coördinates of a point on C^j are given in the several intervals, $t_i^j \leqq t \leqq t_{i+1}^j$, by linear equations

$$x = \frac{t - t_i^j}{t_{i+1}^j - t_i^j} [\varphi(t_{i+1}^j) - \varphi(t_i^j)] = \varphi^j(t),$$

$$y = \frac{t - t_i^j}{t_{i+1}^j - t_i^j} [\psi(t_{i+1}^j) - \psi(t_i^j)] = \psi^j(t).$$

Now

$$\lim_{j=\infty} \varphi^j(t) = \varphi(t), \quad \lim_{j=\infty} \psi^j(t) = \psi(t)$$

uniformly. Hence the sequence of continuous functions of t, $Q[\varphi^j(t), \psi^j(t)]$ approaches $Q[\varphi(t), \psi(t)]$ uniformly as $j = \infty$.

In the sequence of integrals

$$\int_a^b Q[\varphi^j(t), \psi^j(t)]\, d\psi^j(t)$$

$\psi^j(t)$ is of uniformly limited variation in t, $[a \leqq t \leqq b]$. Hence, by a known theorem on the Stieltjes integral†

$$\lim_{j=\infty} \int_a^b Q[\varphi^j(t),\ \psi^j(t)]\, d\psi^j(t) = \int_a^b Q[\varphi(t),\ \psi(t)]\, d\psi(t),$$

or

$$\lim_{j=\infty} \int_{C^j} Q(x,y)\, dy = \int_C Q(x,y)\, dy.$$

We have thus proved that

$$\int\int_S \frac{\partial Q}{\partial x}\, dx\, dy = \int_C Q\, dy.$$

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* Goursat, Cours d'analyse, vol. 1, Ch. 6, p. 126.

† H. E. Bray, These Annals, vol. 20 (1919), p. 180.

3. **Generalizations.** We will next consider a case in which Q is not assumed to be continuous; consequently we will regard the second member of (1) as a Lebesgue-Stieltjes integral.

Let $g(x, y)$ be a summable function in the fundamental rectangle R and $Q(x, y)$ a bounded summable function possessing the property that at every point of R

$$Q(x, y) = \lim_{\mu=0} \frac{1}{\pi\mu^2} \int\int_{S_\mu} Q(x+\xi, y+\eta)\, d\xi\, d\eta,$$

where S_μ is the interior of the circle $\xi^2+\eta^2 = \mu^2$. In other words Q is the symmetric derivative of its own superficial integral, and is therefore the limit of a sequence of continuous functions, as we shall see.

We are now in a position to state:

THEOREM 2. *A necessary and sufficient condition that the equation*

$$\int\int_S g(x, y)\, dx\, dy = \int_C Q(x, y)\, dy$$

be satisfied for every simple rectifiable closed curve C interior to R is that it be satisfied for every circle interior to R.

We have merely to prove the sufficiency of the condition since its necessity is obvious. If instead of g and Q we consider their approximating functions,*

$$g^\mu(x, y) = \frac{1}{\pi\mu^2} \int\int_{S_\mu} g(x+\xi, y+\eta)\, d\xi\, d\eta,$$

$$Q^\mu(x, y) = \frac{1}{\pi\mu^2} \int\int_{S_\mu} Q(x+\xi, y+\eta)\, d\xi\, d\eta,$$

we see that Q^μ is a continuous function, absolutely continuous in x for every y. Now for a given value of y we have, except possibly for values of x which constitute a linear set of zero measure,

$$\frac{\partial}{\partial x} Q^\mu = \frac{1}{\pi\mu^2} \int_{C_\mu} Q(x+\xi, y+\eta)\, d\eta$$

where C_μ is the contour of the circle S_μ. Therefore

$$\frac{\partial}{\partial x} Q^\mu = \frac{1}{\pi\mu^2} \int\int_{S_\mu} g(x+\xi, y+\eta)\, d\xi\, d\eta = g^\mu(x, y)$$

* The main properties of approximating functions of this type are given in the article "Proof of a Formula for an Area", Bull. Amer. Math. Soc., vol. 29 (1923), p. 264.

and, since Q^μ is absolutely continuous in x and g^μ is continuous, this equation is true *for every value of* x *and every value of* y.

We are now able to make use of Theorem 1 which now applies to Q^μ and $g^\mu = \frac{\partial}{\partial x} Q^\mu$. We have

$$\int\int_S g^\mu (x, y)\, dx\, dy = \int_C Q^\mu (x, y)\, dy$$

for every simple rectifiable closed curve C interior to R. If now we regard the quantity $\int_C Q(x, y)\, dy$ as a Lebesgue-Stieltjes integral, we find that

$$\lim_{\mu=0} \int_C Q^\mu (x, y)\, dy = \int_C Q(x, y)\, dy,$$

for Q is bounded and therefore, by definition, the functions Q^μ form a sequence of continuous functions, uniformly bounded.

Again

$$\lim_{\mu=0} \int\int_S g^\mu (x, y)\, dx\, dy = \int\int_S g(x, y)\, dx\, dy$$

because the sequence of integrals

$$\int\int g^\mu (x, y)\, dx\, dy$$

possesses uniform absolute continuity* and g^μ approaches g nearly everywhere, by a well known property of Lebesgue integrals. Therefore

$$\int\int_S g(x, y)\, dx\, dy = \int_C Q(x, y)\, dy,$$

and our theorem is proved.

In connection with this theorem it is interesting to consider the following definitions which are related to those given by Evans.†

Definition 1. Let s be a circle with center at (x_1, y_1), and let c be its circumference and σ its area. Also let α represent an arbitrary direction and α' the direction 90^0 in advance of α. Then, if it exists for every direction α, the quantity

$$D_\alpha Q(x_1, y_1) = \lim_{\sigma=0} \frac{1}{\sigma} \int_C Q(x, y)\, d\alpha'$$

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* de la Vallee Poussin, Trans. Amer. Math. Soc., vol. 29 (1915), p. 445.

† G. C. Evans, Rice Institute Pamphlet, vol. 7 (1920), p. 274-277.

is called the *generalized derivative of Q in the direction α*. It is evident from its definition that $D_\alpha Q$ defines a vector of which it is the component in the direction α. This vector is called the *gradient vector of Q*.

Definition 2. Suppose that $\varphi_\alpha(x, y)$ is summable and is such that for every circle s interior to R the equation

$$\int\int_s \varphi_\alpha \, dx \, dy = \int Q \, d\alpha'$$

is valid, then Q is said to be a potential function for the vector φ. Evidently φ possesses nearly everywhere a gradient vector which is equal nearly everywhere to φ.

In defining the above terms Evans considers a wide class of rectifiable curves instead of circles. To distinguish between our definitions and those of Evans we may refer to ours as "broad" and to those of Evans as "restricted".

The theorem which we have just proved enables us to state the following as a corollary:

Given $Q(x, y)$, a bounded and summable function which is everywhere equal to the superficial derivative of its own integral and which possesses nearly everywhere a summable generalized derivative $D_\alpha Q$ in the broad sense, then a necessary and sufficient condition that Q be a potential function for its generalized derivative in the restricted sense is that the same be true in the broad sense.

So far we have assumed that Q is bounded. It is not easy to formulate Green's lemma, in the case where Q becomes infinite on C, without introducing artificial restrictions. The following theorem, however, is less artificial than it may appear since it applies to the important Stieltjes potential function

$$\Phi(M) = \int_E \log \frac{1}{\overline{MP}} \, df(e)$$

as we shall show.

THEOREM 3. *If Q is summable superficially and is the symmetric derivative of its own superficial integral and if at every point of the rectifiable curve C, $\{Q^\mu(x, y)\}$ is a non decreasing sequence of functions as $\mu = 0$, where, as usual,*

$$Q^\mu(x, y) = \frac{1}{\pi\mu^2} \int\int_{S_\mu} Q(x+\xi, y+\eta) \, d\xi \, d\eta,$$

and if the Lebesgue-Stieltjes integral, $\int_C Q|dy|$, exists, then a sufficient condition that the equation,

$$\int\int_S g(x, y)\, dx\, dy = \int_C Q\, dy,$$

be valid is that it be satisfied for every circle interior to R.

The proof is immediate. As in Theorem 2

$$\frac{\partial}{\partial x} Q^\mu = g^\mu$$

everywhere. Consequently, by Theorem 1

$$\int\int_S g^\mu(x, y)\, dx\, dy = \int_C Q^\mu(x, y)\, dy.$$

As before

$$\lim_{\mu=0} \int\int_S g^\mu(x, y)\, dx\, dy = \int\int_S g(x, y)\, dx\, dy.$$

The fact that

$$\lim_{\mu=0} \int_C Q^\mu(x, y)\, dy = \int_C Q(x, y)\, dy$$

follows from a theorem of P. J. Daniell* on the general integral which states that if, (i) f_n is a non decreasing sequence of functions approaching f as a limit, (ii) the generalized integral $S(f_n)$ exists and (iii) $\lim_{n=\infty} I(f_n)$ exists where I is the associated modular integral, then f is summable (S) and

$$S(f) = \lim_{n=\infty} S(f_n).$$

Here $S(f_n)$ corresponds to $\int Q^\mu(x, y)\, dy$ and $I(f_n)$ corresponds to $\int Q^\mu(x, y)\, |dy|$. The conditions of Daniell's theorem are satisfied, and our theorem is therefore proved.

As an example illustrating the scope of the last theorem consider the function defined by the Stieltjes integral

$$\Phi(x, y) = \int_E \log \frac{1}{r}\, df(e)$$

where f is an additive function of point sets e, of limited variation in the set E. We may assume that f is non negative and bounded since it

* P. J. Daniell, These Annals, vol. 19 (1918), p. 292.

is expressible as the difference of two functions of that kind. Here $r = \sqrt{(x-x')^2+(y-y')^2}$ where (x', y') represents a point of E. Let

$$\psi(x, y) = \int_E \frac{\cos xr}{r} df(e).$$

As Evans has shown,* for any curve C for which the contour integral

$$\int \log \frac{1}{r} |dy|$$

exists, consequently for any circle,

$$\int_C \Phi(x, y) dy = \int\int_S \psi(x, y)\, dx\, dy.$$

Moreover $\lim_{\mu=0} \Phi^\mu = \Phi$ everywhere, for

$$\Phi^\mu(x, y) = \frac{1}{\pi\mu^2}\int\int_{S_\mu} d\xi\, d\eta \int_E \log\frac{1}{r'} df(e) = \frac{1}{\pi\mu^2}\int_E df(e) \int_{S_\mu} \log\frac{1}{r'} d\xi\, d\eta$$

where $r' = \sqrt{(x+\xi-x')^2+(y+\eta-y')^2}$. But

$$\frac{1}{\pi\mu^2}\int\int_{S_\mu} \log\frac{1}{r'} d\xi\, d\eta = \log\frac{1}{r}, \text{ when } r' \geqq \mu,$$

$$= \log\frac{1}{\mu} + \frac{1}{2}\left(1 - \frac{r^2}{\mu^2}\right) \leqq \log\frac{1}{r}, \text{ when } r' \leqq \mu,$$

and these equations define a non-decreasing sequence of continuous functions having $\log\frac{1}{r}$ as limit as $\mu \doteq 0$. Hence $\{\Phi^\mu(x, y)\}$ is a non-decreasing sequence as $\mu \doteq 0$ approaching Φ as a limit wherever Φ exists and becoming positively infinite where Φ is infinite. Consequently, if C is a curve for which $\int_C \log\frac{1}{r} |dy|$ exists, we can state that

$$\int_C \Phi\, dy = \int_S \psi\, dx\, dy.$$

What can be said in the case where C is a general rectifiable curve? By Theorem 1

$$\int_C \Phi^\mu dy = \int_S \psi^\mu\, dx dy$$

* G. C. Evans, loc. cit., p. 264, 3. 2.

for any simple rectifiable curve, and, since ψ is summable (Lebesgue), we have as before

$$\lim_{\mu=\infty} \int\int_S \psi^\mu \, dx\, dy = \int\int_S \psi \, dx\, dy.$$

Now $\Phi(x, y)$ is summable superficially; and being the limit, as we have shown, of a sequence of continuous functions $\{\Phi^\mu\}$ it is measurable in the three dimensional region defined by the variables (ξ, η, s) where s is the arc length along C and $\Phi(x+\xi, y+\eta)$ may be regarded, for the moment, as a function of (ξ, η, s), x and y being functions of s. Consequently

$$\int_C |dy| \int\int_{S_\mu} d\xi\, d\eta \int_E \log \frac{1}{r'}\, df(e) \leq \int_C ds \int\int_{S_\mu} d\xi\, d\eta \int \log \frac{1}{r'}\, df(e)$$
$$= \int\int_{S_\mu} d\xi\, d\eta \int_C ds \int_E \log \frac{1}{r'}\, df(e).$$

Hence the function $\int_C ds \int_E \log \frac{1}{r'}\, df(e)$ of (ξ, η) *exists for nearly every displacement* (ξ, η) *of the curve* C *and so also does*

$$\int_C |dy| \int_E \log \frac{1}{r}\, df(e).$$

Theorem 3 can now be applied and we can state as a theorem:

THEOREM 4. *If C is a rectifiable curve, then for nearly every position, interior to R, into which it can be brought by translation, the equation*

$$\int_C dy \int_E \log \frac{1}{r}\, df(e) = \int_S dx\, dy \int_E \frac{\cos xr}{r}\, df(e)$$

is valid.

It will be noted that, since the right hand member is a Lebesgue integral, it is absolutely continuous as a function of point sets. Consequently the left hand member, regarded as a function of displacements (ξ, η) for a given C, has unnecessary discontinuities. The proof just given shows that for *every* simple rectifiable curve C

$$\lim_{\mu=0} \int_C dy \int_E \left(\log \frac{1}{r}\right)^\mu df(e)$$

exists, and is equal to

$$\int\int_S dx\, dy \int_E \frac{\cos xr}{r}\, df(e).$$

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A CLASS OF MINIMUM PROBLEMS AND THE LINEAR INDEPENDENCE OF FUNCTIONS OF ONE VARIABLE.

BY OTTO DUNKEL.

1. **Introduction.** In two earlier papers the writer gave simple solutions of two problems of the calculus of variations by a method which was mainly algebraic*. In the first paper it was pointed out that the method admitted a generalization by considering any number of auxiliary conditions of a general form. This generalization follows easily along the lines adopted in the two papers, but it may be of interest to develop this general form of the problem in order to show its relation to the criterion for the linear independence of a set of merely continuous functions of a single variable. All of the developments may be applied in the same way to vectors with a finite or infinite number of components, replacing the integrals by inner products; and in a later paper this application will be given together with some additional results. Either form of development leads to a simple proof of Hadamard's Determinant Theorem. The minimum problem which is first considered leads at once to the development of the necessary and sufficient conditions for the linear independence of a set of continuous functions. These conditions will also be obtained by an independent consideration, and from these conditions the solution of the minimum problem will be again obtained. The theorems concerning the variations of the end conditions which were given in the above mentioned papers will be obtained for this general case, and some additional results will be added. A slight generalization of the minimum problem will be considered and it will be pointed out that this generalization includes as a particular case the solution of the problem treated recently by Professor Rider.† An example will be considered in which one of the auxiliary conditions is that the length of the curves considered shall be given. This example includes the cases of the minimum area between a curve and its evolute and the minimum area between a curve and its caustic.

* A determination of the curve minimizing the area enclosed by it and its evolute. The Amer. Math. Monthly, vol. 28 (1921) pp. 15–19, 1921.

A direct determination of the minimum area between a curve and its caustic, Annals of Math., vol. 23 (1921), pp. 135–140. In order to study the minimizing curve this problem was also treated by the processes of the calculus of variation in an article entitled, The curve which with its caustic encloses the minimum area, in the Washington University Studies, vol. 8 (1921), Sci. ser., no. 2, pp. 183–194.

† Rider, On the minimizing of a class of definitive integrals, Annals of Mathematics, 2nd ser. vol. 24 (1922), pp. 167–174.

2. **The minimum integral.** Let $f_1(x)$, $f_2(x)$, $\cdots$, $f_n(x)$ be n continuous, single-valued functions of x in the interval $x_1 \leqq x \leqq x_2$, and let $f(x)$ be any other such function. If we set

$$\int_{x_1}^{x_2} f(x) f_i(x)\, dx = a_i, \qquad i = 1, 2, \cdots, n, \tag{1}$$

where the a_i's are constants determined by these $n+1$ functions, then any function $f(x)$ which satisfies these n integral equations shall be said to belong to the class $C(f, f_1, f_2, \cdots, f_n)$ for the interval $x_1 x_2$. A condition will be placed upon the functions $f_i(x)$ later on. Here we consider the class C as determined by the $n+1$ functions by means of the equations (1), but it will appear later on that the n constants a_i may be arbitrarily assigned and that an infinite number of functions $f(x)$ may be constructed to satisfy the relations (1). The class may then be defined by means of the n functions $f_i(x)$ and the n arbitrarily assigned constants a_i. It will be assumed that all the functions considered are continuous and single-valued also in the enlarged intervals which are considered later. It will be convenient to designate the integral in (1) as the inner product of $f(x)$ and $f_i(x)$ and to use the following symbolic expression for the integral:

$$(f f_i) = (f_i f) = a_i, \qquad i = 1, 2, \cdots, n, \tag{1'}$$

where the limits of integration are omitted for convenience. The function $f(x)$ of the class C which renders the integral

$$\int_{x_1}^{x_2} [f(x)]^2\, dx = (ff) \tag{2}$$

a minimum will now be determined. The integral (2) may be said to be the norm of $f(x)$. If $A_1, A_2, \cdots, A_n$ are constants the integral in (2) may be written

$$\begin{aligned} &\int_{x_1}^{x_2} [f(x)]^2 dx \\ &= \int_{x_1}^{x_2} \Big[f(x) - \sum_1^n A_i f_i(x)\Big]^2 dx + 2\sum_1^n A_i a_i - \int_{x_1}^{x_2} \Big[\sum_1^n A_i f_i(x)\Big]^2 dx. \end{aligned} \tag{2'}$$

It will be shown later that there is a unique determination of the A_i's, under the condition which is mentioned later, so that

$$f_0(x) = \sum_1^n A_i f_i(x) \tag{3}$$

is a member of the class C. With this determination of the constants A_i it is obvious from (2′) that there is one and only one function of the class C, $f_0(x)$, which renders (2) a minimum, and that this minimum is

$$(4) \qquad (f_0 f_0) = \int_{x_1}^{x_2} [f_0(x)]^2\, dx = \sum_1^n A_i a_i.$$

By setting the expression for $f_0(x)$ in the equations (1) or (1′) we obtain n equations for the n constants A_i

$$(5) \quad (f_0 f_j) = a_j = \sum_{i=1}^{i=n} A_i \int_{x_1}^{x_2} f_i(x) f_j(x)\, dx = \sum_{i=1}^{i=n} A_i (f_j f_i), \quad j = 1, 2, \cdots, n,$$

as conditions that $f_0(x)$ shall belong to the class C. If then the determinant of these equations

$$(6) \qquad \begin{vmatrix} (f_1 f_1) & (f_1 f_2) & \cdots & (f_1 f_n) \\ (f_2 f_1) & (f_2 f_2) & \cdots & (f_2 f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (f_n f_1) & (f_n f_2) & \cdots & (f_n f_n) \end{vmatrix} = |f_1 f_2 \cdots f_n|$$

is not zero there is a unique determination of the A_i's, and there is only one function $f_0(x)$ which yields a minimum. It will be assumed then that the determinant in (6) is not zero; this is the condition referred to above in the definition of the class C. The significance of this condition will be considered later. Writing $f(x) - f_0(x)$ as $\Delta f(x)$ the equation (2′) may be written.

$$(7) \qquad (ff) = (\Delta f\, \Delta f) + (f_0 f_0) \quad \text{or} \quad \Delta(ff) = (\Delta f\, \Delta f),$$

where $(\Delta f\, \Delta f) > 0$ if $f(x) \neq f_0(x)$.

The equation (3) with the n equations in (5) may be regarded as $n+1$ homogeneous linear equations in $n+1$ unknowns which are satisfied by $-1, A_1, A_2, \cdots, A_n$. Hence their determinant must be zero. This determinant may then be written as the sum of two determinants, and we obtain finally, using the notation in (6),

$$(8) \qquad |f_1 f_2 \cdots f_n| f_0(x) = -\begin{vmatrix} 0 & f_1(x) & f_2(x) & \cdots & f_n(x) \\ a_1 & (f_1 f_1) & (f_1 f_2) & \cdots & (f_1 f_n) \\ a_2 & (f_2 f_1) & (f_2 f_2) & \cdots & (f_2 f_n) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_n & (f_n f_1) & (f_n f_2) & \cdots & (f_n f_n) \end{vmatrix},$$

which determines $f_0(x)$ completely.

The minimum integral may be obtained in exactly the same way from the equations (4) and (5) by regarding these as $n+1$ homogeneous linear equations. Hence

$$(9)\quad (f_0 f_0) = \int_{x_1}^{x_2} [f_0(x)]^2\, dx = -\frac{\begin{vmatrix} 0 & a_1 & a_2 & \cdots & a_n \\ a_1 & (f_1 f_1) & (f_1 f_2) & \cdots & (f_1 f_n) \\ a_2 & (f_2 f_1) & (f_2 f_2) & \cdots & (f_2 f_n) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_n & (f_n f_1) & (f_n f_2) & \cdots & (f_n f_n) \end{vmatrix}}{|f_1 f_2 \cdots f_n|},$$

where the denominator is defined in (6).

This shows that, if $a_1, a_2, \cdots, a_n$ are any quantities and if

$$|f_1 f_2 \cdots f_n| \neq 0,$$

$$(10)\quad -\begin{vmatrix} 0 & a_1 & a_2 & \cdots & a_n \\ a_1 & (f_1 f_1) & (f_1 f_2) & \cdots & (f_1 f_n) \\ a_2 & (f_2 f_1) & (f_2 f_2) & \cdots & (f_2 f_n) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_n & (f_n f_1) & (f_n f_2) & \cdots & (f_n f_n) \end{vmatrix} |f_1 f_2 \cdots f_n|$$

$$= \int_{x_1}^{x_2} \begin{vmatrix} 0 & f_1(x) & f_2(x) & \cdots & f_n(x) \\ a_1 & (f_1 f_1) & (f_1 f_2) & \cdots & (f_1 f_n) \\ a_2 & (f_2 f_1) & (f_2 f_2) & \cdots & (f_2 f_n) \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ a_n & (f_n f_1) & (f_n f_2) & \cdots & (f_n f_n) \end{vmatrix}^2 dx.$$

If all the A_i's are zero, then all the a_i's must be zero, and conversely, since $|f_1 f_2 \cdots f_n| \neq 0$. Hence, if not all the a_i's are zero, then not all the A_i's can be zero. Also no $f_i(x)$ can be identically zero in the interval $x_1 x_2$, since the determinant (6) is not zero. Thus $f_0(x)$ is identically zero if, and only if, all the a_i's are zero.

3. **Linear independence of a set of continuous functions of one variable.** The significance of the non-vanishing of the determinant (6) will now be considered. It will be shown that if $f(x), f_1(x), f_2(x), \cdots, f_n(x)$ are $n+1$ single-valued and continuous functions of x in the interval $x_1 x_2$, which are linearly independent, then the determinant for these functions

constructed as in (6) is not only different from zero, but is actually greater than zero. These $n+1$ functions are said to be linearly dependent if $n+1$ constants can be found, at least one of which is not zero, such that

$$A_0 f(x) + A_1 f_1(x) + A_2 f_2(x) + \cdots + A_n f_n(x) = 0.$$

If it is impossible to find such a set of constants, then the $n+1$ functions are said to be linearly independent in the given interval. It is clear that if the set of $n+1$ functions are linearly independent then no smaller set selected from them can be linearly dependent: in particular, no function can be identically zero in the interval. Let us assume that the theorem has been proved for n functions, and let us suppose that $f(x), f_1(x), f_2(x), \cdots, f_n(x)$ are $n+1$ linearly independent functions; then, since $f_1(x), f_2(x), \cdots, f_n(x)$ are linearly independent, the determinant (6) is not zero by assumption, and we have a class C of functions determined by these n functions with $f(x)$. To this class there belongs a single function $f_0(x)$ which renders (ff) a minimum, and the equation (7) holds true. Using (4) and (1′) we may write this equation in the form

$$(7') \qquad (ff) - (\Delta f\, \Delta f) = \sum_1^n A_i a_i = \sum_1^n A_i (f_i f).$$

Writing the equations (5) in the form

$$(f_j f) = \sum_1^n A_i (f_j f_i), \qquad j = 1, 2, \cdots, n,$$

we may regard these n equations with (7′) as $n+1$ homogeneous linear equations which are satisfied by $-1, A_1, A_2, \cdots, A_n$, and hence their determinant must be zero. This determinant may be separated into two determinants, and we obtain the result

$$(11) \qquad |ff_1 f_2 \cdots f_n| = (\Delta f\, \Delta f)\, |f_1 f_2 \cdots f_n|.$$

Since $f(x)$ is not linearly dependent upon the remaining functions $f^i(x)$, $\Delta f(x)$ is not identically zero, and hence $(\Delta f\, \Delta f) > 0$. Hence the determinant on the left is greater than zero. To complete the proof we have merely to examine the case of two functions which are linearly independent, say $f_{n-1}(x)$ and $f_n(x)$. In this case we have

$$|f_{n-1} f_n| = (\Delta f_{n-1}\, \Delta f_{n-1})\, (f_n f_n).$$

Here both factors on the right are greater than zero as a consequence of our hypothesis of the linear independence of the two functions, and

hence the left side must be greater than zero. This suffices to complete the proof. If the $n+1$ functions are linearly dependent the determinant on the left in (11) may be written so as to have a column or row repeated or consisting of zero elements, and hence it is zero. The necessary and sufficient condition that the n functions $f_1(x), f_2(x), \cdots, f_n(x)$ shall be linearly independent is that the determinant in (6) shall be greater than zero.

In the equation (11) let us replace $(\Delta f \Delta f)$ by its equivalent expression as given in (7), then

$$|ff_1f_2\cdots f_n| = [(ff)-(f_0f_0)]\,|f_1f_2\cdots f_n|.$$

Now the determinant on the right may be written in a similar manner

$$|f_1f_2\cdots f_n| = [(f_1f_1)-(f_0'f_0')]\,|f_2f_3\cdots f_n|,$$

where $f_0'(x)$ is the minimizing function of the class of functions determined by $f_2(x), f_3(x), \cdots f_n(x)$ with $f_1(x)$. In this case the conditions (1′) are replaced by

$$(f_if_1) = a_i', \qquad i = 2, 3, \cdots, n.$$

By continuing in this way we obtain finally

$$(11') \quad \begin{vmatrix} (f\,f) & (f\,f_1) & \cdots & (f\,f_n) \\ (f_1f) & (f_1f_1) & \cdots & (f_1f_n) \\ (f_2f) & (f_2f_1) & \cdots & (f_2f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (f_nf) & (f_nf_1) & \cdots & (f_nf_n) \end{vmatrix} = [(ff)-(f_0f_0)]\,[(f_1f_1)-(f_0'f_0')]\cdots \cdots [(f_{n-1}f_{n-1})-(f_0^{(n-1)}f_0^{(n-1)})]\,(f_nf_n) \leqq (ff)\,(f_1f_1)\,(f_2f_2)\cdots(f_nf_n).$$

The last result follows from the fact that $(f_0^{(i)}f_0^{(i)}) \geqq 0$ and that $(f_if_i)-(f_0^{(i)}f_0^{(i)}) > 0$, since we have supposed that the $n+1$ functions are linearly independent. If $(f_0^{(i)}f_0^{(i)}) = 0$, then $f_0^{(i)}(x)$ must be identically zero. Hence $(f_if_{i+1}) = (f_if_{i+2}) = \cdots = (f_if_n) = 0$ (see page 290 under (10)). Thus, if the equality signs holds true, the determinant must have all its elements zero except those in the principal diagonal. In this case the $n+1$ functions are said to be orthogonal to each other. If they are not orthogonal the inequality sign must be used. This is essentially Hadamard's Determinant Theorem. It is not difficult to deduce the ordinary form of that theorem from the above result, but this will not be done here since later on in another paper these developments will be applied to vectors and then the theorem will appear in the usual form.

4. The variation of the determinant as the limits of the interval change. It will now be shown that the determinant of the inner products (6) never decreases, but in general increases, as the upper limit x_2 increases or as the lower limit x_1 decreases. To show this in the case that the upper limit x_2 alone changes, we shall write the derivative of the determinant (6) with respect to x_2. This derivative is the sum of the determinants in which each column of (6) is replaced in turn by the column of the derivatives of the elements of that column. To simplify the writing set $f_i(x_2) = c_i$. Then

$$\frac{d|f_1 f_2 \cdots f_n|}{dx_2} = \sum_{j=1}^{j=n} \begin{vmatrix} (f_1 f_1) & \cdots & (f_1 f_{j-1}) & c_1 c_j & (f_1 f_{j+1}) & \cdots & (f_1 f_n) \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ (f_n f_1) & \cdots & (f_n f_{j-1}) & c_n c_j & (f_n f_{j+1}) & \cdots & (f_n f_n) \end{vmatrix}$$

$$= -\sum_{1}^{n} \begin{vmatrix} 0 & 0 & & 0 & c_j & 0 & & 0 \\ c_1 & (f_1 f_1) & \cdots & (f_1 f_{j-1}) & 0 & (f_1 f_{j+1}) & \cdots & (f_1 f_n) \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdot \\ c_n & (f_n f_1) & \cdots & (f_n f_{j-1}) & 0 & (f_n f_{j+1}) & \cdots & (f_n f_n) \end{vmatrix} \tag{12}$$

$$= -\begin{vmatrix} 0 & c_1 & \cdots & c_n \\ c_1 & (f_1 f_1) & \cdots & (f_1 f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_n & (f_n f_1) & \cdots & (f_n f_n) \end{vmatrix}.$$

If we compare this last result with (9) we see that it may be written as $|f_1 f_2 \cdots f_n|\,(g_0 g_0)$ where $g_0(x)$ is the minimizing function for the case in which a_j is replaced by c_j. The expression for $g_0(x)$ is obtained from (8) by replacing the a's by the c's. The result may be written thus

$$\frac{d|f_1 f_2 \cdots f_n|}{dx_2} = \frac{1}{|f_1 f_2 \cdots f_n|} \int_{x_1}^{x_2} \begin{vmatrix} 0 & f_1(x) & \cdots & f_n(x) \\ f_1(x_2) & (f_1 f_1) & \cdots & (f_1 f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_n(x_2) & (f_n f_2) & \cdots & (f_n f_n) \end{vmatrix}^2 dx. \tag{12'}$$

We suppose that the n functions are linearly independent in the interval $x_1 x_2$, or that $|f_1 f_2 \cdots f_n| > 0$. The derivative is therefore never negative; it is zero if, and only if, $f_1(x_2) = f_2(x_2) = \cdots = f_n(x_2) = 0$ (see page 290).

Therefore $|f_1 f_2 \cdots f_n|$ increases when x_2 increases unless all the functions $f_i(x)$ are zero throughout the interval of increase. The same analysis applies to the case of the change of the lower limit x_1 with the result stated above. The case in which the n functions are linearly dependent offers no difficulty.

5. **The minimum integral as a function of the limits of integration.** It will now be shown that as x_2 increases or as x_1 decreases the minimum integral (4) never increases and in general decreases, if the a_i's remain constant. Suppose first that x_1 remains fixed while x_2 varies. In the expression for the minimum integral (4) the A_i's alone vary with x_2; hence

$$\frac{d(f_0 f_0)}{dx_2} = \sum_1^n a_i \frac{dA_i}{dx_2}. \tag{13}$$

By differentiating the equations (5) for a_j we obtain a set of equations

$$\sum_{i=1}^{i=n} (f_j f_i) \frac{dA_i}{dx_2} + f_j(x_2) \sum_{i=1}^{i=n} A_i f_i(x_2) = 0, \quad j = 1, 2, \cdots, n. \tag{14}$$

Multiply (14) by A_j and take the sum of the products from $j = 1$ to $j = n$ and there results

$$\sum_{i=1}^{i=n} \sum_{j=1}^{j=n} A_j (f_j f_i) \frac{dA_i}{dx_2} = -\sum_{j=1}^{j=n} \sum_{i=1}^{i=n} A_j A_i f_j(x_2) f_i(x_2).$$

This reduces by the use of (5) and (3) to

$$\sum_1^n a_i \frac{dA_i}{dx_2} = -[f_0(x_2)]^2.$$

Combining this result with (13) we obtain the final form of the derivative

$$\frac{d(f_0 f_0)}{dx_2} = -[f_0(x_2)]^2. \tag{15}$$

This shows that as x_2 increases $(f_0 f_0)$ decreases unless $f_0(x)$ is zero throughout the interval of increase of x_2. A similar procedure would show that

$$\frac{d(f_0 f_0)}{dx_1} = [f_0(x_1)]^2.$$

It follows then that as x_1 decreases the minimum integral decreases in general.

Suppose now that $f'(x)$ is any function of x such that

$$\int_{x_1}^{x_2'} f'(x) f_i(x)\, dx = \int_{x_1}^{x_2} f_0(x) f_i(x)\, dx = a_i, \quad i = 1, 2, \cdots, n, \tag{16}$$

where $f_0(x)$ has the same meaning as before, i. e.,

$$f_0(x) = \sum_1^n A_i f_i(x).$$

Suppose that $x_2' < x_2$. If we assume that the n given functions $f_1(x)$, $f_2(x), \cdots, f_n(x)$ are linearly independent in the interval $x_1 x_2'$, that the ends of this interval are the limits of integration in the corresponding expressions for the inner products, and that $f_0'(x)$ is the minimizing function for this shorter interval; then $(f'f')$ is greater than $(f_0'f_0')$, unless $f'(x) = f_0'(x)$. It follows from the proof above that $(f_0 f_0)$ is less than $(f_0' f_0')$, unless $f_0(x)$ is zero in the interval $x_2' x_2$ and it may then be shown that $f_0'(x) = f_0(x)$ in the interval $x_1 x_2'$. Hence

$$\int_{x_1}^{x_2'} [f'(x)]^2\, dx \geq \int_{x_1}^{x_2} [f_0(x)]^2\, dx, \qquad x_2' < x_2, \tag{17}$$

the equality sign holding only if $f'(x) = f_0(x)$ in the interval $x_1 x_2'$ and if $f_0(x) = 0$ in $x_2' x_2$.

This may also be shown without the assumption of the linear independence of the n given functions in the interval $x_1 x_2'$. Multiply both sides of (16) by A_i and add the resulting integrals on each side, we obtain in this way

$$\int_{x_1}^{x_2'} f'(x) f_0(x)\, dx = \int_{x_1}^{x_2} [f_0(x)]^2\, dx. \tag{18}$$

This enables us to write, after a cancellation by use of (18),

$$\begin{aligned} &\int_{x_1}^{x_2'} [f'(x)]^2\, dx - \int_{x_1}^{x_2} [f_0(x)]^2\, dx \\ &\qquad = \int_{x_1}^{x_2'} [f'(x) - f_0(x)]^2\, dx + \int_{x_2'}^{x_2} [f_0(x)]^2\, dx. \end{aligned} \tag{19}$$

We infer from this result the same facts as before.

If $x_2' > x_2$ we may again use (18) and write

$$\int_{x_1}^{x_2'} [f'(x)]^2\,dx - \int_{x_1}^{x_2} [f_0(x)]^2\,dx$$

$$= \int_{x_1}^{x_2} [f'(x) - f_0(x)]^2\,dx + \int_{x_2}^{x_2'} [f'(x)]^2\,dx - 2\int_{x_2}^{x_2'} f'(x) f_0(x)\,dx. \tag{20}$$

Hence if $f'(x)$ and $f_0(x)$ have opposite signs in the interval $x_2 x_2'$ the difference on the left is greater than zero.

Consider again the interval $x_1 x_2'$. If $x_2' > x_2$ then the n given functions being linearly independent in $x_1\, x_2$ must necessarily be linearly independent in the larger interval $x_1\, x_2'$; if $x_2' < x_2$ we shall assume that the n functions are linearly independent in the smaller interval. Suppose as before that $f_0'(x)$ is the minimizing function for the interval $x_1\, x_2'$. We have the relation (18) in which $f'(x)$ is replaced by $f_0'(x)$ and a second relation obtained in a similar manner

$$\int_{x_1}^{x_2'} [f_0'(x)]^2\,dx = \int_{x_1}^{x_2} f_0(x) f_0'(x)\,dx. \tag{18'}$$

From these two equations we have at once

$$\int_{x_1}^{x_2'} [f_0'(x)]^2\,dx - \int_{x_1}^{x_2} [f_0(x)]^2\,dx = -\int_{x_2}^{x_2'} f_0'(x)\, f_0(x)\,dx. \tag{21}$$

This formula may be used to obtain again the formula (15) by the reasoning used in a previous paper.*

The variation of the lower limit x_1 may be studied in an entirely similar manner.

6. **The addition of auxiliary conditions.** In addition to the n auxiliary conditions in (1) let us impose a further condition of the same type

$$(f f_{n+1}) = \int_{x_1}^{x_2} f(x) f_{n+1}(x)\,dx = a_{n+1}, \tag{22}$$

such that the $n+1$ functions $f_1(x)$, $f_2(x)$, $\cdots$, $f_{n+1}(x)$ are linearly independent. We may denote the class of functions $f(x)$ which satisfy the new demands by C_{n+1}, the former class being now designated by C_n. There is one and only one function of the class C_{n+1} which renders (2) a minimum and we shall denote it by $f_0^{(n+1)}(x)$. This notation has a different meaning from that on page 292. The minimizing function for C_n

* Annals of Math., l. c., p. 139.

shall here be denoted by $f_0^{(n)}(x)$. Since $f_0^{(n+1)}(x)$ is also a member of the class C_n we must have

$$(f_0^{(n+1)} f_0^{(n+1)}) \geqq (f_0^{(n)} f_0^{(n)}). \tag{23}$$

The equality sign holds only if $f_0^{(n+1)}(x) = f_0^{(n)}(x)$ and then from (22) we have

$$(f_0^{(n)} f_{n+1}) = a_{n+1}.$$

On the other hand if this is true, then $f_0^{(n)}(x) = f_0^{(n+1)}(x)$, since there is only one linear combination of the $n+1$ functions which satisfy the $n+1$ auxiliary conditions. From (8) we have

$$\begin{vmatrix} f_0^{(n)}(x) & f_1(x) & \cdots & f_n(x) \\ a_1 & (f_1 f_1) & \cdots & f_1 f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & (f_n f_1) & \cdots & f_n f_n) \end{vmatrix} = 0;$$

and, if we multiply this equation by $f_{n+1}(x)$ and integrate, we obtain as as the necessary and sufficient conditions for the equality in (23)

$$\begin{vmatrix} a_1 & (f_1 f_1) & \cdots & (f_1 f_n) \\ a_2 & (f_2 f_1) & \cdots & (f_2 f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n+1} & (f_{n+1} f_1) & \cdots & (f_{n+1} f_n) \end{vmatrix} = 0. \tag{24}$$

This would suggest that $(f_0^{(n+1)} f_0^{(n+1)}) - (f_0^{(n)} f_0^{(n)})$ is a function of the determinant to the left in (24) which is always positive. In order to determine this function we may use the two sets of equations

$$\begin{aligned} f_0^{(n+1)}(x) &= \sum_1^{n+1} A_i^{(n+1)} f_i(x), \quad f_0^{(n)}(x) = \sum_1^{n} A_i^{(n)} f_i(x), \\ a_j &= \sum_1^{n+1} A_i^{(n+1)} (f_j f_i), \quad a_j = \sum_1^{n} A_i^{(n)} (f_j f_i), \qquad j = 1, 2, \cdots, n, \\ a_{n+1} &= \sum_1^{n+1} A_i^{(n+1)} (f_{n+1} f_i). \end{aligned} \tag{25}$$

The value of $A_{n+1}^{(n+1)}$ is obtained by solving the first set of $n+1$ equations above, omitting the equation for $f_0^{(n+1)}(x)$, and it is found to be

$$(26)\qquad A_{n+1}^{(n+1)} = \frac{\begin{vmatrix} (f_1 f_1) & \cdots & (f_1 f_n) & a_1 \\ (f_2 f_1) & \cdots & (f_2 f_n) & a_2 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ (f_{n+1} f_1) & \cdots & (f_{n+1} f_n) & a_{n+1} \end{vmatrix}}{|f_1 f_2 \cdots f_{n+1}|}.$$

If we subtract corresponding equations in (25) we may write the resulting equations as

$$f_0^{(n+1)}(x) - f_0^{(n)}(x) - A_{n+1}^{(n+1)} f_{n+1}(x) = \sum_1^n B_i f_i(x)$$

$$-A_{n+1}^{(n+1)}(f_j f_{n+1}) = \sum_1^n B_i (f_j f_i), \quad j = 1, 2, \cdots, n.$$

Considering these as $n+1$ homogeneous equations which have the solution $-1, B_1, B_2, \cdots, B_n$, where $A_{n+1}^{(n+1)}$ is regarded as known, the determinant of the equations must be zero. This determinant breaks up into two determinants one of which is equal to $|f_1 f_2 \cdots f_n|\,[f_0^{(n+1)}(x) - f_0^{(n)}(x)]$ and the other is equal to $-A_{n+1}^{(n+1)}$ times the numerator in (26) after replacing a_i by $f_i(x)$. We obtain thus

$$(27)\qquad f_0^{n+1}(x) - f_0^{(n)}(x) = \frac{\begin{vmatrix} (f_1 f_1) & \cdots & (f_1 f_n) & a_1 \\ (f_2 f_1) & \cdots & (f_2 f_n) & a_2 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ (f_{n+1} f_1) & \cdots & (f_{n+1} f_n) & a_{n+1} \end{vmatrix} \Big| \begin{vmatrix} (f_1 f_1) & \cdots & (f_1 f_n) & f_1(x) \\ (f_2 f_1) & \cdots & (f_2 f_n) & f_2(x) \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ (f_{n+1} f_1) & \cdots & (f_{n+1} f_n) & f_{n+1}(x) \end{vmatrix}}{|f_1 f_2 \cdots f_n| \,\Big|\, |f_1 f_2 \cdots f_{n+1}|}$$

By the same method used on pages 295, 296 to obtain (18) and (18′) we find

$$(f_0^{(n)} f_0^{(n)}) = (f_0^{(n)} f_0^{(n+1)}),$$

and this gives

$$(28')\qquad \int_{x_1}^{x_2} [f_0^{(n+1)}(x)]^2 dx - \int_{x_1}^{x_2} [f_0^{(n)}(x)]^2 dx = \int_{x_1}^{x_2} [f_0^{(n+1)}(x) - f_0^{(n)}(x)]^2 dx$$

$$= \int_{x_1}^{x_2} f_0^{(n+1)}(x)\,[f_0^{(n+1)}(x) - f_0^{(n)}(x)]\, dx.$$

Inserting in this last result the value of $f_0^{(n+1)}(x) - f_0^{(n)}(x)$ as given in (27) we have finally

$$(28)\qquad (f_0^{(n+1)} f_0^{(n+1)}) - (f_0^{(n)} f_0^{(n)}) = \frac{\begin{vmatrix} (f_1 f_1) & \cdots & (f_1 f_n) & a_1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ (f_{n+1} f_1) & \cdots & (f_{n+1} f_n) & a_{n+1} \end{vmatrix}^2}{|f_1 f_2 \cdots f_n|\,|f_1 f_2 \cdots f_{n+1}|}.$$

Hence the minimum increases when a new condition is introduced unless the numerator in the expression above is zero.

7. **The reverse procedure.** All of these results may be obtained by starting from the theorem of linear independence, which will now be established by an independent process. Suppose that $f(x)$, $f_1(x), \cdots, f_n(x)$ are $n+1$ continuous and single-valued functions, and consider the determinant

$$(29)\qquad \begin{vmatrix} (ff) & (ff_1) & \cdots & (ff_n) \\ (f_1 f) & (f_1 f_1) & \cdots & (f_1 f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (f_n f) & (f_n f_1) & \cdots & (f_n f_n) \end{vmatrix} = |f f_1 \cdots f_n|.$$

If the $n+1$ functions are linearly dependent then the determinant is easily seen to be zero. Since the variable of integration has no effect upon the final result provided the limits x_1, x_2 remain the same in each of the integrals $(f_i f_j)$, we may suppose that in the first column the variable is t, in the second, $t_1, \cdots$, in the last, t_n. Then (29) may be written as a multiple integral

$$(30')\qquad \int_{x_1}^{x_2}\int_{x_1}^{x_2}\cdots\int_{x_1}^{x_2} \begin{vmatrix} f(t) & f(t_1) & \cdots & f(t_n) \\ f_1(t) & f_1(t_1) & \cdots & f_1(t_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_n(t) & f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix} f(t) f_1(t_1)\cdots f_n(t_n)\, dt\, dt_1 \cdots dt_n.$$

Now the result is the same if in (30′) we interchange t, t_1, $\cdots$, t_n in any way. If, for example, we carry out the substitution

$$\begin{pmatrix} t & t_1 & t_2 & \cdots & t_n \\ t_{i_0} & t_{i_1} & t_{i_2} & \cdots & t_{i_n} \end{pmatrix}$$

we obtain in the integrand the same determinant with the sign given by the substitution, + if the substitution is even, — if it is odd, and the factor

$$f(t_{i_0}) f_1(t_{i_1}) \cdots f_n(t_{i_n}).$$

But this factor with the sign thrown off by the determinant is a term of the same determinant. Hence adding all of these $(n+1)!$ equal expressions for (29) and dividing the sum by $(n+1)!$ we have as the value of (29)

$$(30)\quad \frac{1}{(n+1)!}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\cdots\int_{x_1}^{x_2}\begin{vmatrix} f(t) & f(t_1) & \cdots & f(t_n) \\ f_1(t) & f_1(t_1) & \cdots & f_1(t_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_n(t) & f_n(t_1) & \cdots & f_n(t_n) \end{vmatrix}^2 dt\,dt_1\cdots dt_n.$$

Hence the determinant in (29) is never negative and it is zero only if the determinant in the integrand of (30) vanishes for every set of values for $t, t_1, \cdots, t_n$ in the region of integration $x_1 \leqq t_i \leqq x_2$, $i = 0, 1, \cdots, n$. This follows from the fact that the $n+1$ functions are continuous and hence the integrand in (30) is a continuous function of the $n+1$ variables t_i. Suppose then (29) is zero and hence (30) is zero. Now (30) may be zero as a result of some one or more of the functions $f_i(x)$ being identically zero. In this case the set of functions are linearly dependent. If none of the elements are identically zero, then, since the determinant itself is identically zero, there must be some minor which is not identically zero whereas all minors of a higher order are identically zero. Let then such a minor be the minor of order r in the upper left-hand corner of the determinant in (30), and let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be values of $t, t_1, \cdots, t_{r-1}$ for which it is not zero, and A_r the value of this minor for the special set of arguments. Then the determinant of the $(r+1)$st order in the upper left-hand corner is identically zero, and hence it is zero when $t, t_1 \cdots, t_{r-1}$ have the values $\alpha_1, \alpha_2, \cdots, \alpha_r$, respectively, and t_r has any value in the interval $x_1\, x_2$. Developing this determinant in terms of the elements of the $(r+1)$st column and replacing t_r by t, we have

$$A_0 f(t) + A_1 f_1(t) + \cdots + A_r f_r(t) = 0,$$

where the A_i's are minors of the rth order with the proper sign, and constants. Since A_r is not zero and t may have any value in the given interval, the set of $n+1$ functions are linearly dependent. It has now been shown that if (29) is zero the $n+1$ functions are linearly dependent and conversely. The necessary and sufficient condition that the functions be linearly independent is that (29) be greater than zero.

Before taking up the minimum problem we shall work out another form for (29). Consider the integral

$$\int_{x_1}^{x_2} \begin{vmatrix} f(t) & f_1(t) & \cdots & f_n(t) \\ (f_1 f) & (f_1 f_1) & \cdots & (f_1 f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (f_n f) & (f_n f_1) & \cdots & (f_n f_n) \end{vmatrix} \begin{vmatrix} 0 & 0 & & f_i(t) & & 0 \\ (f_1 f) & (f_1 f_1) & \cdots & (f_1 f_i) & \cdots & (f_1 f_n) \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ (f_n f) & (f_n f_1) & \cdots & (f_n f_i) & \cdots & (f_n f_n) \end{vmatrix} dt.$$

The second factor in the integrand is $f_i(t)$ times a constant, and if the factor $f_i(t)$ be multiplied into the first row of the first determinant factor, the result of the integration will be zero if $i = 1, 2, \cdots, n$, since then a row of the determinant (29) is repeated. If $f_i(t)$ stands in the first column, in which case it is $f(t)$, the result of the integration is $|ff_1 \cdots f_n| \, |f_1 f_2 \cdots f_n|$. If then we take the sum of these $n+1$ integrals we have

$$(31) \qquad |f_1 f_2 \cdots f_n| \, |ff_1 \cdots f_n| = \int_{x_1}^{x_2} \begin{vmatrix} f(t) & f_1(t) & \cdots & f_n(t) \\ (f_1 f) & (f_1 f_1) & \cdots & (f_1 f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (f_n f) & (f_n f_1) & \cdots & (f_n f_n) \end{vmatrix}^2 dt.$$

Let us now turn to the minimum problem, and let us suppose that $f_1(x), f_2(x), \cdots, f_n(x)$ are linearly independent functions. If we set as before

$$(ff_i) = a_i, \qquad i = 1, 2, \cdots, n,$$

then we have a class of functions C in the interval $x_1 x_2$ determined as before by the n equations above. The equation given by (29) and (30), or the equation in (31), may be written

$$|f_1 f_2 \cdots f_n| \, (ff) = -\begin{vmatrix} 0 & a_1 & \cdots & a_n \\ a_1 & (f_1 f_1) & \cdots & (f_1 f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & (f_n f_1) & \cdots & (f_n f_n) \end{vmatrix} + \int_{x_1}^{x_2} [\delta f(t)]^2 \, dt$$

in which the only variables are (ff) and $\delta f(t)$. Hence the minimum of (ff) is obtained if, and only if, $\delta f(t)$ is zero, and this means that $f(x)$ is linearly dependent upon $f_1(x), f_2(x), \cdots, f_n(x)$, since these n functions are linearly independent. The minimum value of (ff) is easily read off from

the above. The function $f_0(x)$ which yields the minimum is most easily obtained from (31), for it must make the integrand on the right zero. Hence, after inserting the values of $(ff_i) = a_i$, we have

$$(8') \qquad \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_n(x) \\ a_1 & (f_1 f_1) & \cdots & (f_1 f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & (f_n f_1) & \cdots & (f_n f_n) \end{vmatrix} = 0$$

and this determines $f_0(x)$ completely.

The expression (30) gives a second proof of the fact that (29) never decreases as x_2 increases or as x_1 decreases.

8. **The class C.** It has been shown in the previous proofs that the class C contains the function $f_0^{(n)}(x)$, or as sometimes denoted $f_0(x)$, which is linearly dependent upon the given functions $f_1(x), f_2(x), \cdots, f_n(x)$. It will now be shown that if the set of constants $a_1, a_2, \cdots, a_n$ be chosen arbitrarily an infinite number of functions $f(x)$ can be found which satisfy the relations (1). For this purpose consider the determinant

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_{n+1} \\ f_1(t_1) & f_1(t_2) & \cdots & f_1(t_{n+1}) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_n(t_1) & f_n(t_2) & \cdots & f_n(t_{n+1}) \end{vmatrix} = \sum_1^{n+1} y_i F_i,$$

where $t_1, t_2, \cdots, t_{n+1}$ are unequal values of x in the interval $x_1 x_2$. Since we suppose that the n functions $f_1(x), f_2(x), \cdots, f_n(x)$ are linearly independent it is possible to choose the arguments $t_1, t_2, \cdots, t_{n+1}$ in such a way that not all the first minors of the above determinant, after omitting the row of y's, are zero. This follows directly from the proofs on page 300. Denote the cofactors of the first row by F_i, and let us suppose that F_1 is not zero. Then the above determinant is, for the selected values of the arguments t_i, a linear homogeneous polynomial in the y's which is not identically zero. It is therefore possible to choose values for the y's in many ways so that the determinant is not zero. With such a set of values for the y's it is possible in many ways to construct a single-valued and continuous function of x, $f_{n+1}(x)$, which for $x = t_1, t_2, \cdots, t_{n+1}$ takes on the values $y_1, y_2, \cdots, y_{n+1}$, respectively. Hence this function $f_{n+1}(x)$ is

linearly independent of the given set of n functions by the argument on page 300.

Now adjoin to the conditions (1) the condition

$$(ff_{n+1}) = a_{n+1},$$

where $f_{n+1}(x)$ is any function of x which is linearly independent of the n given functions and which may be determined as above. The constant a_{n+1}, is to be chosen such that the determinant (24) page 297 is not zero. There is only one value of a_{n+1} which makes the determinant zero, since in (24) a_{n+1} is the only unknown and its coefficient in the linear polynomial on the left is not zero. There are then with a given $f_{n+1}(x)$ an infinite number of determinations of a_{n+1} such that the determinant (24) is not zero.

It has been shown that there is a function $f_0^{(n+1)}(x)$ which is linearly dependent upon $f_1(x), f_2(x), \cdots, f_{n+1}(x)$ and which satisfies the n conditions of class C together with the adjoined condition which is mentioned above. It has been shown that this function must be different from $f_0^{(n)}(x)$. Thus $f_0^{(n+1)}(x)$ is a member of class C which is linearly independent of $f_1(x)$, $f_2(x), \cdots, f_n(x)$ and it admits of an infinite number of determinations varying with the choice of $f_{n+1}(x)$ and of a_{n+1}.

9. **A simplified form of the minimum problem.** If the functions $f_1(x), f_2(x), \cdots, f_n(x)$ are orthogonal to one another, that is if

$$(f_i f_j) = 0, \qquad i \neq j,$$

the results will appear in a much simpler form, and some of the proofs may be simplified. If the given functions are not orthogonal they may be replaced by linear combinations of n linearly independent functions $\psi_i(x)$ which are orthogonal to one another, and an equivalent minimum problem may be stated in terms of the new functions. The function $f_0^{(n+1)}(x) - f_0^{(n)}(x)$ on page 298 is orthogonal to $f_1(x), f_2(x), \cdots, f_n(x)$, and we could build up according to this type the desired set of orthogonal functions. We shall proceed, however, in a slightly different manner. In the conditions (1) set

$$a_1 = \sqrt{\frac{|f_1 f_2 \cdots f_n|}{|f_2 f_3 \cdots f_n|}}, \qquad a_i = 0, \qquad i = 2, 3, \cdots, n.$$

If we denote the minimizing function in this case by $\psi_1(x)$ then

$$(\psi_1 f_i) = 0, \qquad i = 2, 3, \cdots, n.$$

The function $\psi_1(x)$ is orthogonal not only to each of the functions $f_i(x)$ with a subscript different from 1 but also to any linear combination of these $n-1$ functions with constant coefficients. We have from (8) after a slight reduction

$$\psi_1(x) = \frac{\begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ (f_2 f_1) & (f_2 f_2) & \cdots & (f_2 f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (f_n f_1) & (f_n f_2) & \cdots & (f_n f_n) \end{vmatrix}}{\sqrt{|f_1 f_2 \cdots f_n|\,|f_2 f_3 \cdots f_n|}} \tag{32}$$

From (9) we have also

$$(\psi_1\,\psi_1) = 1. \tag{32'}$$

Taking now the functions $f_2(x), f_3(x), \cdots, f_n(x)$ we proceed in exactly the same way to construct

$$\psi_2(x) = \frac{\begin{vmatrix} f_2(x) & f_3(x) & \cdots & f_n(x) \\ (f_3 f_2) & (f_3 f_3) & \cdots & (f_3 f_n) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ (f_n f_2) & (f_n f_3) & \cdots & (f_n f_n) \end{vmatrix}}{\sqrt{|f_2 f_3 \cdots f_n|\,|f_3 f_4 \cdots f_n|}};$$

and finally

$$\psi_n(x) = \frac{f_n(x)}{\sqrt{(f_n f_n)}}.$$

It follows from the construction of these function that

$$\begin{aligned} (\psi_i\,\psi_j) &= 0, && i \neq j, \\ &= 1, && i = j. \end{aligned}$$

The equations of transformation are of the form

$$\psi_i(x) = A_{ii} f_i(x) + \cdots + A_{in} f_n(x), \qquad A_{ii} = \sqrt{\frac{|f_{i+1} f_{i+2} \cdots f_n|}{|f_i f_{i+1} \cdots f_n|}},$$

and their determinant is $|f_1 f_2 \cdots f_n|^{-1/2}$. This determinant is clearly not zero. Hence if we set

$$A_{ii} a_i + \cdots + A_{in} a_n = b_i, \qquad i = 1, 2, \cdots, n,$$

we shall have

$$(\psi_i f) = b_i, \qquad i = 1, 2, \cdots, n.$$

Every function $f(x)$ which satisfies these equations must satisfy the original set of equations (1) and conversely.

Since $f_0(x)$ is linearly dependent upon the functions $\psi_i(x)$ we have

$$f_0(x) = \sum_1^n B_i \psi_i(x),$$

and, if we multiply both sides of this equation by $\psi_i(x)$ and integrate, we obtain $(f_0 \psi_i) = b_i = B_i$. Hence

$$f_0(x) = \sum_1^n b_i \psi_i(x).$$

Multiplying both sides of this equation by $f_0(x)$ and integrating, we find

$$(f_0 f_0) = \sum_1^n b_i^2.$$

10. **A modified form of the minimum problem.** A minimum problem will now be stated which is only apparently a generalization of the preceeding one. We shall consider a set of functions $F(x)$, $\varphi(x)$, $F_1(x)$, $F_2(x), \cdots, F_n(x)$ each of which is single-valued and continuous in the interval $x_1 \leqq x \leqq x_2$. The last n functions are supposed to be given and are linearly independent; $\varphi(x)$ is supposed to be a given function greater than zero throughout the interval. The first function $F(x)$ is any function of the class determined by the equations

$$a_i = \int_{x_1}^{x_2} F(x) F_i(x) dx, \qquad i = 1, 2, \cdots, n, \tag{33}$$

where the a_i's are constants. The problem is to find that function of the class which renders the integral

$$\int_{x_1}^{x_2} \varphi(x) [F(x)]^2 dx \tag{34}$$

a minimum.

If we set $f_i(x) = F_i(x) / \sqrt{\varphi(x)}$ and $f(x) = F(x) \sqrt{\varphi(x)}$, it will be seen at once that if the functions $F_i(x)$ are linearly dependent, the functions $f_i(x)$ are also linearly dependent, and conversely, since $\varphi(x) > 0$.

Since the functions $F_i(x)$ are linearly independent, the functions $f_i(x)$ must also be linearly independent. Hence this substitution carries this problem into the one previously treated. There is therefore one and only one solution

$$(35) \qquad F_0(x) = \frac{f_0(x)}{\sqrt{\varphi(x)}} = \frac{A_1 F_1(x) + A_2 F_2(x) + \cdots + A_n F_n(x)}{\varphi(x)}$$

where the A's are uniquely determined in the original problem.

There are certain problems of the calculus of variations which these results solve. The two problems which have been treated by the writer and to which reference has been made at the beginning of this paper may be considered as special case of this class of problems. In a recent article Professor Rider has treated a form of integral which includes these two problems. Professor Rider's* problem may be stated as follows: To find the curve $y = f(x)$ which renders the integral

$$(36) \qquad \int_0^{x_2} \frac{[1+y'^2]^m}{y''}\,dx, \qquad y' = \frac{dy}{dx}, \qquad y'' = \frac{d^2y}{dx^2},$$

a minimum, where $y'' > 0$ in the interval of integration, subject to the conditions that the curve passes through the two fixed points (0,0), (x_2, y_2) with given slopes $p_1 = \tan\tau_1$, $p_2 = \tan\tau_2$, $p_1 < p_2$ or $\tau_1 < \tau_2$ at the respective end points. This problem may thrown into the form treated here by introducing the angle τ as independent variable. Setting $y' = \tan\tau$ the integral becomes

$$(37) \qquad \int_{\tau_1}^{\tau_2} (\sec\tau)^{2m-4} R^2\,d\tau = 4\int_{\tau_1}^{\tau_2} (\sec\tau)^{2m-2} \delta^2\,d\tau,$$

where R is the radius of curvature of the curve expressed as a function of τ, and δ is the length of the ray to the caustic of the curve.† Either form may be treated with the same ease. We shall consider the first, and in this case the end conditions are

$$(38) \qquad x_2 = \int_{\tau_1}^{\tau_2} R\cos\tau\,d\tau, \qquad y_2 = \int_{\tau_1}^{\tau_2} R\sin\tau\,d\tau.$$

The functions $\cos\tau$ and $\sin\tau$ are obviously linearly independent. The solution may be written at once and certain facts concerning the variations of the end conditions may be stated from the preceding theory.

* Rider, Annals of Math. l. c.

† Annals of Math., vol. 28 (1921), p. 135.

It was pointed out in the caustic problem that in case a cusp appears between the ends of the minimizing curve the treatment in the non-parametric form may not be valid.* This difficulty is avoided by the introduction of the parameter τ. The restrictions placed upon m by Professor Rider are not needed in this treatment of the necessary and sufficient conditions for a minimum. We may, in fact, replace $[1+y'^2]^m$ in (36) by any single-valued and continuous function of y' which does not vanish, or we may replace the integral (37) by the more general form

$$\int_{\tau_1}^{\tau_2} \varphi(\tau) R^2 \, d\tau \tag{39}$$

where $\varphi(\tau)$ is single-valued and continuous and does not vanish in the interval of integration. The end conditions are given in (38). There is a unique determination of the minimizing function $R_0(\tau)$ and it may be written at once by means of (8) page 289 and (35) page 306. It has the form

$$R_0(\tau) = \frac{A\cos\tau + B\sin\tau}{\varphi(\tau)} \tag{40}$$

where A and B are certain constants which may be obtained from the general expression (8). We have then by quadrature

$$x = \int_{\tau_1}^{\tau} R_0(\tau)\cos\tau \, d\tau, \qquad y = \int_{\tau_1}^{\tau} R_0(\tau)\sin\tau \, d\tau. \tag{41}$$

If the constants A and B are such that $R_0(\tau)$ vanishes between τ_1 and τ_2 then there is a cusp between the ends of the minimizing curve. If B is not zero, the minimizing function $f(x)$ becomes multipled valued.

Suppose that we impose in addition to the conditions in (38) the requirement that

$$l = \int_{\tau_1}^{\tau_2} R \, d\tau \tag{42}$$

where l is any constant. Here again the functions 1, $\cos\tau$, $\sin\tau$, are linearly independent, and there exists a unique solution, whatever the assigned value of l may be, in the form

$$R_0(\tau) = \frac{A\cos\tau + B\sin\tau + C}{\varphi(\tau)}$$

* Dunkel, Washington University Studies, l. c., p. 193.

where A, B, C are constants which are easily written out. The equations of the curve which furnish the minimum are obtained as in (41). If $R_0(\tau)$ does not change sign between τ_1 and τ_2, then l is the length of the curve between the end points. In this case we must have $l^2 > x_2^2 + y_2^2$. If, for example, $A^2 + B^2$ turns out to be less than or equal to C^2 this will be the case. If $l^2 \leqq x_2^2 + y_2^2$ then R_0 must change sign and there will be one or more cusps in the minimizing curve between the ends. If, for example, there is only one cusp, then l is the difference in the lengths of the parts of the curve before and after the cusp. As an extreme case, if $l = 0$ and there is only one cusp between the ends, it bisects the length of the curve.

If we drop any of the conditions in (38) and (42) the minimum is reduced in general (page 299). If we drop, for example, the conditions (38) which require the curve to pass through (x_2, y_2) and retain only (42), then there can be no change in sign of R, since then

$$R_0(\tau) = \frac{C}{\varphi(\tau)}, \qquad C = \frac{l}{\int_{\tau_1}^{\tau_2} \frac{d\tau}{\varphi(\tau)}}.$$

If in the above $\varphi(\tau) = 1$, we have the problem of the minimum area between a curve and its evolute; if $\varphi(\tau) = \cos^2 \tau/4$, we have the problem of the minimum area between a curve and its caustic. The general theory which has been developed gives a number of facts in each case regarding the variation of the end conditions.

TABLES OF QUADRATIC FORMS.

By Albert E. Cooper.

The following tables of the binary quadratic forms having for determinants the negative numbers from —101 to —200 are arranged according to the notations given in the *Disquisitiones arithmeticae* and followed in turn by A. Caylay in his compilation published in Crelle, LX (1862). The tables are an extension of the latter ones. There is exhibited representations of all the classes of forms of numbers from —101 to —200, their quadratic characters, supplementary characters, and a table of composition for these forms.

D	Class	a	b	c	d	e	de	Cp
−101	1. 0.101	+			+			1
	9. 4. 13	+			+			o^2
	5.–2. 21	+			+			o^4
	6. 1. 17	+			+			o^6
	6.–1. 17	+			+			o^8
	5. 2. 21	+			+			o^{10}
	9.–*4. 13*	+			+			o^{12}
	3. 1. 34	−			−			o
	10.–3. 11	−			−			o^3
	7. 2. 15	−			−			o^5
	2. 1. 51	−			−			o^7
	7.–2. 15	−			−			o^9
	10. 3. 11	−			−			o^{11}
	3.–1. 34	−			−			o^{13}
−102	1. 0.102	+	+			+		1
	3. 0. 34	+	−			−		c_1
	2. 0. 51	−	+			−		c_2
	6. 0. 17	−	−			+		c_1c_2
−103	1. 0.103	+						1
	7. 3. 16	+						f
	8.–1. 13	+						f^2
	8. 1. 13	+						f^3
	7.–*3. 16*	+						f^4
	2. 1. 52	+						σ
	8.–3. 14	+						σf
	4.–1. 26	+						σf^2
	4. 1. 26	+						σf^3
	8. 3. 14	+						σf^4

D	Class	a	b	c	d	e	de	Cp
−104	1. 0.104	+			+	+		1
	9.–2. 12	+			+	+		g^2
	9. *2. 12*	+			+	+		g^4
	3. 1. 35	+			−	−		g
	4. 2. 27	+			−	−		g^3
	3.–*1. 35*	+			−	−		g^5
	5.–1. 21	−			+	−		cg^2
	8. 0. 13	−			+	−		c
	5. *1. 21*	−			+	−		cg^4
	7.–1. 15	−			−	+		cg^5
	8. 4. 15	−			−	+		cg^3
	7. 1. 15	−			−	+		cg
−105	1. 0.105	+	+	+	+			1
	11. 4. 11	−	+	+	−			c_1
	3. 0. 35	−	−	−	−			c_2
	2. 1. 53	−	−	+	+			c_3
	10. 5. 13	+	−	−	+			c_1c_2
	7. 0. 15	+	−	+	−			c_1c_3
	6. 3. 19	+	+	−	−			c_2c_3
	5. 0. 21	−	+	−	+			$c_1c_2c_3$
−106	1. 0.106	+					+	1
	10.–2. 11	+					+	g^2
	10. *2. 11*	+					+	g^4
	5. 2. 22	−					−	g
	2. 0. 53	−					−	g^3
	5.–2. 22	−					−	g^5

D	Class	a	b	c	d	e	de	Cp
−107	1. 0.107	+						1
	3. 1. 36	+						j
	9. 1. 12	+						j^2
	4.–1. 27	+						j^3
	11. 5. 12	+						j^4
	11.–5. 12	+						j^5
	4. 1. 27	+						j^6
	9.–1. 12	+						j^7
	3.–1. 36	+						j^8
	2. 1. 54	+						σ
	6. 1. 18	+						σj
	6.–1. 18	+						σj^2
−108	1. 0.108	+			+			1
	9.–3. 13	+			+			g^2
	9. *3. 13*	+			+			g^4
	7. 2. 16	+			−			g
	4. 0. 27	+			−			g^3
	7.–2. 16	+			−			g^5
−109	1. 0.109	+			+			1
	5. 1. 22	+			+			g^2
	10.–1. 11	−			−			g
	2. 1. 55	−			−			g^3
	10. 1. 11	−			−			g^5
	5.–1. 22	+			+			g^4

D	Class	a	b	c	d	e	de	Cp
−110	1. 0.110	+	+			+		1
	9. 4. 14	+	+			+		g^2
	9.–*4. 14*	+	+			+		g^4
	3. 1. 37	−	+			−		g
	5. 0. 22	−	+			−		g^3
	3.–*1. 37*	−	+			−		g^5
	7. 3. 17	−	−			+		cg^2
	2. 0. 55	−	−			+		c
	7.–*3. 17*	−	−			+		cg^4
	6. 2. 19	+	−			−		cg^5
	10. 0. 11	+	−			−		cg^3
	6.–2. 19	+	−			−		cg
−111	1. 0.111	+	+					1
	7. 1. 16	+	+					i^2
	3. 0. 37	+	+					i^4
	7.–*1. 16*	+	+					i^6
	5. 2. 23	−	−					i
	8.–3. 15	−	−					i^3
	8. 3. 15	−	−					i^5
	5.–2. 23	−	−					i^7
	2. 1. 56	+	+					σ
	10.–3. 12	−	−					σi
	8.–1. 14	+	+					σi^2
	4. 1. 28	−	−					σi^3
	6. 3. 20	+	+					σi^4
	4.–1. 28	−	−					σi^5
	8. 1. 14	+	+					σi^6
	10. 3. 12	−	−					σi^7
−112	1. 0.112	+			+	+		1
	11. 3. 11	+			−	−		c_1
	4. 2. 29	+			+	−		c_2
	7. 0. 16	+			−	+		c_1c_2
−113	1. 0.113	+			+			1
	9.–2. 13	+			+			i^2
	2. 1. 57	+			+			i^4
	9. *2. 13*	+			+			i^6
	3. 1. 38	−			−			i
	6.–1. 19	−			−			i^3
	6. 1. 19	−			−			i^5
	3.–1. 38	−			−			i^7
−114	1. 0.114	+	+				+	1
	6. *0. 19*	+	+				+	e^2
	5. 1. 23	−	+				−	e
	5.–*1. 23*	−	+				−	e^3
	10.–4. 13	+	−				−	ce
	10. *4. 13*	+	−				−	ce^3
	2. 0. 57	−	−				+	c
	3. 0. 38	−	−				+	ce^2

D	Class	a	b	c	d	e	de	Cp
−115	1. 0.115	+	+					1
	4.–1. 29	+	+					g^2
	4. *1. 29*	+	+					g^4
	7. 2. 17	−	−					g
	5. 0. 23	−	−					g^3
	7.–2. 17	−	−					g^5
	2. 1. 58	+	+					σ
	10. 5. 14	−	−					σg
−116	1. 0.116	+				+		1
	9. 1. 13	+				+		m^2
	5. 2. 24	+				+		m^4
	4. 0. 29	+				+		m^6
	5.–2. 24	+				+		m^8
	9.–*1. 13*	+				+		m^{10}
	3. 1. 39	−				−		m
	8.–2. 15	−				−		m^3
	11. 4. 12	−				−		m^5
	11.–4. 12	−				−		m^7
	8. 2. 15	−				−		m^9
	3.–1. 39	−				−		m^{11}
−117	1. 0.117	+	+			+		1
	9. *0. 13*	+	+			+		e^2
	7. 3. 18	+	−			−		e
	7.–*3. 18*	+	−			−		e^4
	9.–3. 14	−	+			+		ce
	9. *3. 14*	−	+			+		ce^3
	2. 1. 59	−	−			−		c
	11. 2. 11	−	−			−		ce^2
−118	1. 0.118	+					+	1
	7. 1. 17	+					+	g^2
	7.–*1. 17*	+					+	g^4
	11. 5. 13	−					−	g
	2. 0. 59	−					−	g^3
	11.–5. 13	−					−	g^5

D	Class	a	b	c	d	e	de	Cp
−119	1. 0.119	+	+					1
	9. 4. 15	+	+					k^2
	8.–1. 15	+	+					k^4
	8. 1. 15	+	+					k^6
	9.–*4. 15*	+	+					k^8
	3. 1. 40	−	−					k
	5. 1. 24	−	−					k^3
	7. 0. 17	−	−					k^5
	5.–1. 24	−	−					k^7
	3.–1. 40	−	−					k^9
	2. 1. 60	+	+					σ
	6. 1. 20	−	−					σk
	8.–3. 16	+	+					σk^2
	10. 1. 12	−	−					σk^3
	4.–1. 30	+	+					σk^4
	12. 5. 12	−	−					σk^5
	4. 1. 30	+	+					σk^6
	10.–1. 12	−	−					σk^7
	8. 3. 16	+	+					σk^8
	6.–1. 20	−	−					σk^9
−120	1. 0.120	+	+		+	+		1
	11. 1. 11	−	+		−	−		c_1
	3. 0. 40	+	−		−	−		c_2
	5. 0. 24	−	+		+	−		c_3
	8. 4. 17	−	−		+	+		c_1c_2
	4. 2. 31	+	+		−	+		c_1c_3
	8. 0. 15	−	−		−	+		c_2c_3
	12. 6. 13	+	−		+	−		$c_1c_2c_3$
−121	1. 0.121	+			+			1
	5. 2. 25	+			+			g^2
	5.–*2. 25*	+			+			g^4
	10. 3. 13	−			+			g
	2. 1. 61	−			+			g^3
	10.–3. 13	−			+			g^5
−122	1. 0.122	+					+	1
	9.–2. 14	+					+	k^2
	6. 2. 21	+					+	k^4
	6.–2. 21	+					+	k^6
	9. *2. 14*	+					+	k^8
	3. 1. 41	−					−	k
	7.–2. 18	−					−	k^3
	5. 0. 61	−					−	k^5
	7. 2. 18	−					−	k^7
	3.–1. 41	−					−	k^9
−123	1. 0.123	+	+					1
	4.–1. 31	+	+					g^2
	4. *1. 31*	+	+					g^4
	11. 3. 12	−	−					g
	3. 0. 41	−	−					g^3
	11.–3. 12	−	−					g^5
	2. 1. 62	+	+					σ
	6. 3. 22	−	−					σg

D	Class	a	b	c	d	e	de	Cp
−124	1. 0.124	+			+			1
	5. 1. 25	+			+			g^2
	5.−*1. 25*	+			+			g^4
	7. 3. 19	+			−			g
	4. 0. 31	+			−			g^3
	7.−3. 19	+			−			g^5
−125	1. 0.125	+			+			1
	9. 1. 14	+			+			k^2
	6.−1. 21	+			+			k^4
	6. 1. 21	+			+			k^6
	9.−*1. 14*	+			+			k^8
	3. 1. 43	−			−			k
	7. 1. 18	−			−			k^3
	2. 1. 63	−			−			k^5
	7.−1. 18	−			−			k^7
	3.−1. 43	−			−			k^9
−126	1. 0.126	+	+			+		1
	7. *0. 18*	+	+			+		e^2
	5. 2. 26	−	−			−		e
	5.−*2. 26*	−	−			−		e^3
	10. 2. 13	+	−			−		ce
	10.−*2. 13*	+	−			−		ce^3
	2. 0. 63	−	+			+		c
	9. 0. 14	−	+			+		ce^2
−127	1. 0.127	+						1
	8. 3. 17	+						f
	11. 4. 13	+						f^2
	11.−4. 13	+						f^3
	8.−3. 17	+						f^4
	2. 1. 64	+						σ
	4.−1. 32	+						σf
	8.−1. 16	+						σf^2
	8. 1. 16	+						σf^3
	4. 1. 32	+						σf^4
−128	1. 0.128				+	+		1
	9. 4. 16				+	+		i^2
	4. 2. 33				+	+		i^4
	9.−*4. 16*				+	+		i^6
	3. 1. 43				−	−		i
	11.−2. 12				−	−		i^3
	11. 2. 12				−	−		i^5
	3.−1. 43				−	−		i^7

D	Class	a	b	c	d	e	de	Cp
−129	1. 0.129	+	+		+			1
	10.−1. 13	+	+		+			g^2
	10. *1. 13*	+	+		+			g^4
	5. 1. 26	−	−		+			g
	2. 1. 65	−	−		+			g^3
	5.−*1. 26*	−	−		+			g^5
	7.−2. 19	+	−		−			cg^2
	3. 0. 43	+	−		−			c
	7. *2. 19*	+	−		−			cg^4
	11.−5. 14	−	+		−			cg^5
	6. 3. 23	−	+		−			cg^3
	11. 5. 14	−	+		−			cg
−130	1. 0.130	+	+				+	1
	2. 0. 65	−	−				+	c_1
	5. 0. 26	+	−				−	c_2
	10. 0. 13	−	+				−	c_1c_2
−131	1. 0.131	+						1
	7. 3. 20	+						p^1
	3. 1. 44	+						p^2
	11. 1. 12	+						p^3
	9.−2. 15	+						p^4
	4.−1. 33	+						p^5
	5. 2. 27	+						p^6
	12.−5. 13	+						p^7
	12. 5. 13	+						p^8
	5.−2. 27	+						p^9
	4. 1. 33	+						p^{10}
	9. 2. 15	+						p^{11}
	11.−1. 12	+						p^{12}
	3.−1. 44	+						p^{13}
	7.−3. 20	+						p^{14}
	2. 1. 66	+						σ
	10.−3. 14	+						σp
	6. 1. 22	+						σp^2
	6.−1. 22	+						σp^3
	10. 3. 14	+						σp^4
−132	1. 0.132	+	+		+			1
	4. *0. 33*	+	+		+			e^2
	7. 1. 19	+	−		−			e
	7.−*1. 19*	+	−		−			e^3
	8. 2. 17	−	−		+			ce
	8.−*2. 17*	−	−		+			ce^3
	3. 0. 44	−	+		−			c
	11. 0. 12	−	+		−			ce^2
−133	1. 0.133	+	+		+			1
	2. 1. 67	+	−		−			c_1
	7. 0. 19	−	+		−			c_2
	13. 6. 13	−	−		+			c_1c_2

D	Class	a	b	c	d	e	de	Cp
−134	1. 0.134	+				+		1
	9. 1. 15	+				+		s^2
	10.−4. 15	+				+		s^4
	6. 2. 23	+				+		s^6
	6.−2. 23	+				+		s^8
	10. 4. 15	+				+		s^{10}
	9.−*1. 15*	+				+		s^{12}
	3. 1. 45	−				−		s
	5.−1. 27	−				−		s^3
	11. 3. 13	−				−		s^5
	2. 0. 67	−				−		s^7
	11.−3. 13	−				−		s^9
	5. 1. 27	−				−		s^{11}
	3.−1. 45	−				−		s^{13}
−135	1. 0.135	+	+					1
	9. 3. 16	+	+					g^2
	9.−*3. 16*	+	+					g^4
	8. 1. 17	−	−					g
	5. 0. 27	−	−					g^3
	8.−1. 17	−	−					g^5
	2. 1. 68	+	+					σ
	4. 1. 34	−	−					σg
	8.−3. 18	+	+					σg^2
	10. 5. 16	−	−					σg^3
	8. 3. 18	+	+					σg^4
	4.−1. 34	−	−					σg^5
−136	1. 0.136	+			+	+		1
	8. *0. 17*	+			+	+		e^2
	7. 2. 20	−			−	+		e
	7.−*2. 20*	−			−	+		e^3
	5.−2. 28	−			+	−		ce
	5. *2. 28*	−			+	−		ce^3
	4. 2. 35	+			−	−		c
	8. 4. 19	+			−	−		ce^3
−137	1. 0.137	+			+			1
	9. 4. 17	+			+			i^2
	2. 1. 69	+			+			i^4
	9.−*4. 17*	+			+			i^6
	3. 1. 43	−			−			i
	6.−1. 23	−			−			i^3
	6. 1. 23	−			−			i^5
	3.−1. 43	−			−			i^7
−138	1. 0.138	+	+				+	1
	3. *0. 46*	+	+				+	e^2
	7. 3. 21	+	−				−	e
	7.−*3. 21*	+	−				−	e^3
	11. 4. 14	−	−				+	ce
	11.−*4. 14*	−	−				+	ce^3
	2. 0. 67	−	+				−	c
	6. 0. 23	−	+				−	ce^2

D	Class	a	b	c	d	e	de	Cp
−139	1. 0.139	+						1
	5. 1. 28	+						j
	7. 1. 20	+						j^2
	4.−1. 35	+						j^3
	11.−2. 13	+						j^4
	11. 2. 13	+						j^5
	4. 1. 35	+						j^6
	7.−1. 20	+						j^7
	5.−1. 28	+						j^8
	2. 1. 70	+						σ
	10. 1. 14	+						σj
	10.−1. 14	+						σj^2
−140	1. 0.140	+	+		+			1
	9.−2. 16	+	+		+			g^2
	9. *2. 16*	+	+		+			g^4
	3. 1. 47	−	−		−			g
	7. 0. 20	−	−		−			g^3
	3.−*1. 47*	−	−		−			g^5
	11.−5. 15	+	+		−			cg^2
	4. 0. 35	+	+		−			c
	11. *5. 15*	+	+		−			cg^4
	12.−4. 13	−	−		+			cg^5
	5. 0. 28	−	−		+			cg^3
	12. 4. 13	−	−		+			cg
−141	1. 0.141	+	+		+			1
	6. *3. 25*	+	+		+			e^2
	5. 2. 29	−	−		+			e
	5.−*2. 29*	−	−		+			e^3
	10.−3. 15	+	−		−			ce
	10. *3. 15*	+	−		−			ce^3
	2. 1. 71	−	+		−			c
	3. 0. 47	−	+		−			ce^2
−142	1. 0.142	+				+		1
	2. *0. 71*	+				+		e^2
	11. 1. 13	−				−		e
	11.−1. 13	−				−		e^3

D	Class	a	b	c	d	e	de	Cp
−143	1. 0.143	+	+					1
	9. 1. 16	+	+					k^2
	8.−3. 19	+	+					k^4
	8. 3. 19	+	+					k^6
	9.−*1. 16*	+	+					k^8
	3. 1. 48	−	−					k
	7. 2. 21	−	−					k^3
	11. 0. 13	−	−					k^5
	7.−2. 21	−	−					k^7
	3.−1. 48	−	−					k^9
	2. 1. 72	+	+					σ
	6. 1. 24	−	−					σk
	8.−1. 18	+	+					σk^2
	12. 5. 14	−	−					σk^3
	4. 1. 36	+	+					σk^4
	12. 1. 12	−	−					σk^5
	4.−1. 36	+	+					σk^6
	12.−5. 14	−	−					σk^7
	8. 1. 18	+	+					σk^8
	6.−1. 24	−	−					σk^9
−144	1. 0.144	+			+	+		1
	9. *0. 16*	+			+	+		e^2
	5. 1. 29	−			+	−		e
	5.−*1. 29*	−			+	−		e^3
	9. 3. 17	−			+	+		ce
	9.−*3. 17*	−			+	+		ce^3
	4. 2. 37	+			+	−		c
	13. 5. 13	+			+	−		ce^2
−145	1. 0.145	+	+		+			1
	5. *0. 29*	+	+		+			e^2
	7. 3. 22	−	+		−			e
	7.−*3. 22*	−	+		−			e^3
	11.−3. 14	+	−		−			ce
	11. *3. 14*	+	−		−			ce^3
	2. 1. 73	−	−		+			c
	10. 5. 17	−	−		+			ce^2
−146	1. 0.146	+					+	1
	3.−1. 49	+					+	q^2
	9.−4. 18	+					+	q^4
	6.−2. 25	+					+	q^6
	2. 0. 73	+					+	q^8
	6. 2. 25	+					+	q^{10}
	9. 4. 18	+					+	q^{12}
	3. *1. 49*	+					+	q^{14}
	7. 1. 21	−					−	q
	10. 2. 15	−					−	q^3
	5.−2. 30	−					−	q^5
	13.−6. 14	−					−	q^7
	13. 6. 14	−					−	q^9
	5. 2. 30	−					−	q^{11}
	10.−2. 15	−					−	q^{13}
	7.−1. 21	−					−	q^{15}

D	Class	a	b	c	d	e	de	Cp
−147	1. 0.147	+	+					1
	4. 1. 37	+	+					g^2
	4.−*1. 37*	+	+					g^4
	12. 3. 13	+	−					g
	3. 0. 49	+	−					g^3
	12.−3. 13	+	−					g^5
	2. 1. 74	+	+					σ
	6. 3. 26	+	−					σg
−148	1. 0.148	+			+			1
	8. 2. 19	−			−			e
	4. 0. 37	+			+			e^2
	8.−2. 19	−			−			e^3
−149	1. 0.149	+			+			1
	9.−2. 17	+			+			s^2
	5.−1. 30	+			+			s^4
	6.−1. 25	+			+			s^6
	6. 1. 25	+			+			s^8
	5. 1. 30	+			+			s^{10}
	9. *2. 17*	+			+			s^{12}
	3. 1. 50	−			−			s
	10. 1. 15	−			−			s^3
	11.−4. 15	−			−			s^5
	2. 1. 75	−			−			s^7
	11. 4. 15	−			−			s^9
	10.−1. 15	−			−			s^{11}
	3.−1. 50	−			−			s^{13}
−150	1. 0.150	+	+			+		1
	6. *0. 25*	+	+			+		e^2
	7. 2. 22	+	−			+		e
	7.−*2. 22*	+	−			+		e^3
	11.−2. 14	−	+			−		ce
	11. *2. 14*	−	+			−		ce^3
	2. 0. 75	−	−			−		c
	3. 0. 50	−	−			−		ce^2
−151	1. 0.151	+						1
	5. 2. 31	+						h
	8. 1. 19	+						h^2
	11. 5. 16	+						h^3
	11.−5. 16	+						h^4
	8.−1. 19	+						h^5
	5.−2. 31	+						h^6
	2. 1. 76	+						σ
	10.−3. 16	+						σh
	4. 1. 38	+						σh^2
	8. 3. 20	+						σh^3
	8.−3. 20	+						σh^4
	4.−1. 38	+						σh^5
	10. 3. 16	+						σh^6

D	Class	a	b	c	d	e	de	Cp
−152	1. 0.152	+			+	+		1
	9. 1. 17	+			+	+		g^2
	9.−*1. 17*	+			+	+		g^4
	3. 1. 51	−			−	−		g
	8. 0. 19	−			−	−		g^3
	3.−*1. 51*	−			−	−		g^5
	7.−3. 23	+			−	+		cg^2
	4. 2. 39	+			−	+		c
	7. *3. 23*	+			−	+		cg^4
	12. 2. 13	−			+	−		cg^5
	8. 4. 21	−			+	−		cg^3
	12.−2. 13	−			+	−		cg
−153	1. 0.152	+	+		+			1
	13. *4. 13*	+	+		+			e^2
	7. 1. 22	+	−		−			e
	7.−*1. 22*	+	−		−			e^3
	11.−1. 14	−	−		−			ce
	11. *1. 14*	−	−		−			ce^3
	2. 1. 77	−	+		+			c
	9. 0. 17	−	+		+			ce^2
−154	1. 0.154	+	+				+	1
	11. *0. 14*	+	+				+	e^2
	5. 1. 31	−	+				−	e
	5.−*1. 31*	−	+				−	e^3
	10.−4. 17	−	−				+	ce
	10. *4. 17*	−	−				+	ce^3
	2. 0. 77	+	−				−	c
	7. 0. 22	+	−				−	ce^2
−155	1. 0.155	+	+					1
	2. 4. 19	+	+					m^2
	4.−1. 39	+	+					m^4
	5. 0. 31	+	+					m^6
	4. 1. 39	+	+					m^8
	9.−*4. 19*	+	+					m^{10}
	3. 1. 52	−	−					m
	12.−1. 13	−	−					m^3
	12. 5. 15	−	−					m^5
	12.−5. 15	−	−					m^7
	12. 1. 13	−	−					m^9
	5.−1. 52	−	−					m^{11}
	2. 1. 78	+	+					σ
	6. 1. 26	−	−					σm
	6.−1. 26	−	−					σm^2
	10. 5. 18	+	+					σm^3
−156	1. 0.156	+	+		+			1
	12. *0. 13*	+	+		+			e^2
	5. 2. 32	−	−		+			e
	5.−*2. 32*	−	−		+			e^3
	11. 3. 15	−	−		−			ce
	11.−*3. 15*	−	−		−			ce^3
	3. 0. 52	+	+		−			c
	4. 5. 39	+	+		−			ce^2

D	Class	a	b	c	d	e	de	Cp
−157	1. 0.157	+			+			1
	13.−5. 14	+			+			g^2
	13. *5. 14*	+			+			g^4
	7. 2. 23	−			−			g
	2. 1. 79	−			−			g^3
	7.−2. 23	−			−			g^5
−158	1. 0.158	+				+		1
	9.−2. 18	+				+		i^2
	2. 0. 79	+				+		i^4
	9. *2. 18*	+				+		i^6
	3. 1. 53	−				−		i
	6. 2. 27	−				−		i^3
	6.−2. 27	−				−		i^5
	3.−1. 53	−				−		i^7
−159	1. 0.159	+	+					1
	7.−3. 24	+	+					k^2
	13. 6. 15	+	+					k^4
	13.−6. 15	+	+					k^6
	7. *3. 24*	+	+					k^8
	5. 1. 32	−	−					k
	8.−3. 21	−	−					k^3
	3. 0. 53	−	−					k^5
	8. 3. 21	−	−					k^7
	5.−1. 32	−	−					k^9
	2. 1. 80	+	+					σ
	10. 1. 16	−	−					σk
	12. 3. 14	+	+					σk^2
	4.−1. 40	−	−					σk^3
	8.−1. 20	+	+					σk^4
	6. 3. 28	−	−					σk^5
	8. 1. 20	+	+					σk^6
	4. 1. 40	−	−					σk^7
	12.−3. 14	+	+					σk^8
	10.−1. 16	−	−					σk^9
−160	1. 0.160	+			+	+		1
	4. *2. 41*	+			+	+		e^2
	7. 1. 23	-			−	+		e
	7.−*1. 23*	−			−	+		e^3
	11.−4. 16	+			−	−		ce
	11. *4. 16*	+			−	−		ce^3
	5. 0. 32	−			+	−		c
	13. 3. 13	−			+	−		ce^2

D	Class	a	b	c	d	e	de	Cp
−161	1. 0.161	+	+		+			1
	9. 1. 18	+	+		+			i^2
	2. 1. 81	+	+		+			i^4
	9.−*1. 18*	+	+		+			i^6
	3. 1. 54	−	+		−			i
	6.−1. 27	−	+		−			i^3
	6. 1. 27	−	+		−			i^5
	3.−*1. 54*	−	+		−			i^7
	7. 0. 23	+	−		−			c
	11.−2. 15	+	−		−			ci^2
	14. 7. 15	+	−		−			ci^4
	11. *2. 15*	+	−		−			ci^6
	10. 3. 17	−	−		+			ci
	5. 2. 33	−	−		+			ci^3
	5.−2. 33	−	−		+			ci^5
	10.−3. 17	−	−		+			ci^7
−162	1. 0.162	+					+	1
	9. 3. 19	+					+	g^2
	9.−*3. 19*	+					+	g^4
	11. 5. 17	-					+	g
	2. 0. 81	−					+	g^3
	11.−5. 17	−					+	g^5
−163	1. 0.163	+						1
	4. 1. 41	+						d
	4.−1. 41	+						d^2
	2. 1. 82	+						σ
−164	1. 0.164	+			+			1
	9. 4. 20	+			+			q^2
	8. 2. 21	+			+			q^4
	5.−1. 33	+			+			q^6
	4. 0. 41	+			+			q^8
	5. 1. 33	+			+			q^{10}
	8.−2. 21	+			+			q^{12}
	9.−*4. 20*	+			+			q^{14}
	3. 1. 55	−			-			q
	7.−2. 24	−			−			q^3
	11. 1. 15	−			−			q^5
	12.−4. 15	−			−			q^7
	12. 4. 15	−			−			q^9
	11.−1. 15	−			−			q^{11}
	7. 2. 24	-			−			q^{13}
	3.−1. 55	−			−			q^{15}
−165	1. 0.165	+	+	+	+			1
	13. 2. 13	+	−	−	+			c_1
	3. 0. 55	+	−	+	−			c_2
	2. 1. 83	−	−	−	−			c_3
	10. 5. 19	+	+	−	−			c_1c_2
	11. 0. 15	−	+	+	−			c_1c_3
	6. 3. 29	−	+	−	+			c_2c_3
	5. 0. 33	−	−	+	+			$c_1c_2c_3$

D	Class	a	b	c	d	e	de	Cp
−166	1. 0.166	+				+		1
	7. 3. 25	+				+		k^2
	10.–2. 17	+				+		k^4
	10. 2. 17	+				+		k^6
	7.–*3. 25*	+				+		k^8
	5. 2. 34	—				—		k
	13.–4. 14	—				—		k^3
	2. 0. 83	—				—		k^5
	13. 4. 14	—				—		k^7
	5.–2. 34	—				—		k^9
−167	1. 0.167	+						1
	3. 1. 56	+						l
	9.–2. 19	+						l^2
	8. 1. 21	+						l^3
	7.–1. 24	+						l^4
	11.–3. 16	+						l^5
	11. 3. 16	+						l^6
	7. 1. 24	+						l^7
	8.–1. 21	+						l^8
	9. 2. 19	+						l^9
	3.–1. 56	+						l^{10}
	2. 1. 84	+						σ
	6. 1. 28	+						σl
	12. 5. 16	+						σl^2
	4. 1. 42	+						σl^3
	12. 1. 14	+						σl^4
	8. 3. 22	+						σl^5
	8.–3. 22	+						σl^6
	12.–1. 14	+						σl^7
	4.–1. 42	+						σl^8
	12.–5. 16	+						σl^9
	6.–1. 28	+						σl^{10}
−168	1. 0.168	+	+		+	+		1
	13. 1. 13	+	—		+	—		c_1
	3. 0. 56	—	—		—	—		c_2
	4. 2. 43	+	+		−	—		c_3
	8. 4. 23	—	+		—	+		c_1c_2
	7. 0. 24	+	—		—	+		c_1c_3
	12. 6. 17	—	—		+	+		c_2c_3
	8. 0. 21	—	+		+	—		$c_1c_2c_3$
−169	1. 0.169	+			+			1
	10.–1. 17	+			+			g^2
	10. *1. 17*	+			+			g^4
	5. 1. 34	—			+			g
	2. 1. 85	—			+			g^3
	5.–1. 34	—			+			g^5

D	Class	a	b	c	d	e	de	Cp
−170	1. 0.170	+	+				+	1
	9. 1. 19	+	+				+	g^2
	9.–*1. 19*	+	+				+	g^4
	3. 1. 57	—	—				+	g
	10. 0. 17	—	—				+	g^3
	3.–*1. 57*	—	—				+	g^5
	13.–5. 15	—	+				—	cg^2
	2. 0. 85	—	+				—	c
	13. *5. 15*	—	+				—	cg^4
	6. 2. 29	+	—				—	cg^5
	5. 0. 34	+	—				—	cg^3
	6.–2. 29	+	—				—	cg
−171	1. 0.171	+	+					1
	7.–2. 25	+	+					m^2
	4.–1. 43	+	+					m^4
	9. 0. 19	+	+					m^6
	4. 1. 43	+	+					m^8
	7. *2. 25*	+	+					m^{10}
	5. 2. 35	—	+					m
	9.–3. 20	—	+					m^3
	11. 4. 17	—	+					m^5
	11.–4. 17	—	+					m^7
	9. 3. 20	—	+					m^9
	5.–2. 35	—	+					m^{11}
	2. 1. 86	+	+					σ
	10.–3. 18	—	+					σm
	14. 5. 14	+	+					σm^2
	10. 3. 18	—	+					σm^3
−172	1. 0.172	+			+			1
	13. 6. 16	+			+			g^2
	13.–*6. 16*	+			+			g^4
	11. 2. 16	+			−			g
	4. 0. 43	+			−			g^3
	11.–2. 16	+			−			g^5
−173	1. 0.173	+			+			1
	9. 4. 21	+			+			s^2
	13. 3. 14	+			+			s^4
	6.–1. 29	+			+			s^6
	6. 1. 29	+			+			s^8
	13.–3. 14	+			+			s^{10}
	9.–*4. 21*	+			+			s^{12}
	3. 1. 58	—			—			s
	7. 3. 26	—			—			s^3
	11.–5. 18	—			—			s^5
	2. 1. 87	—			—			s^7
	11. 5. 18	—			—			s^9
	7.–3. 26	—			—			s^{11}
	3.–1. 58	—			—			s^{13}

D	Class	a	b	c	d	e	de	Cp
−174	1. 0.174	+	+			+		1
	7.–1. 25	+	+			+		g^2
	7. *1. 25*	+	+			+		g^4
	5. 1. 35	—	+			—		g
	6. 0. 29	—	+			—		g^3
	5.–*1. 35*	—	+			—		g^5
	14. 6. 15	—	—			+		cg^2
	2. 0. 87	—	—			+		c
	14.–*6. 15*	—	—			+		cg^4
	10. 4. 19	+	—			—		cg^5
	3. 0. 58	+	—			—		cg^3
	10.–4. 19	+	—			—		cg
−175	1. 0.175	+	+					1
	11. 1. 16	+	+					g^2
	11.–*1. 16*	+	+					g^4
	8. 3. 23	—	+					g
	7. 0. 25	—	+					g^3
	8.–3. 23	—	+					g^5
	2. 1. 88	+	+					σ
	4.–1. 44	—	+					σg
	8.–1. 22	+	+					σg^2
	14. 7. 16	—	+					σg^3
	8. 1. 22	+	+					σg^4
	4. 1. 44	—	+					σg^5
−176	1. 0.176	+			+	+		1
	9.–2. 20	+			+	+		g^2
	9. *2. 20*	+			+	+		g^4
	3. 1. 59	+			−	−		g
	11. 0. 16	+			−	−		g^3
	3.–*1. 59*	+			−	−		g^5
	5. 2. 36	+			+	−		cg^2
	4. 2. 45	+			+	−		c
	5.–*2. 36*	+			+	−		cg^4
	12. 2. 15	+			−	+		cg^5
	15. 7. 15	+			−	+		cg^3
	12.–2. 15	+			−	+		cg
−177	1. 0.177	+	+		+			1
	3. 0. 59	—	+		—			c_1
	2. 1. 89	—	−		+			c_2
	6. 3. 31	+	—		−			c_1c_2
−178	1. 0.178	+					+	1
	11. 3. 17	+					+	i^2
	2. 0. 89	+					+	i^4
	11.–*3. 17*	+					+	i^6
	7. 2. 26	—					—	i
	13. 2. 14	—					—	i^3
	13.–2. 14	—					—	i^5
	7.–2. 26	—					—	i^7

D	Class	a	b	c	d	e	de	Cp
−179	1. 0.179	+						1
	5. 1. 36	+						p
	12. 1. 15	+						p^2
	3.–1. 60	+						p^3
	13. 4. 15	+						p^4
	4. 1. 45	+						p^5
	9.–1. 20	+						p^6
	12.–5. 7	+						p^7
	12. 5. 7	+						p^8
	9. 1. 20	+						p^9
	4.–1. 45	+						p^{10}
	13.–4. 15	+						p^{11}
	3. 1. 60	+						p^{12}
	12.–1. 15	+						p^{13}
	5.–1. 36	+						p^{14}
	2. 1. 90	+						σ
	10. 1. 18	+						σp
	6. 1. 30	+						σp^2
	6.–1. 30	+						σp^3
	10.–1. 18	+						σp^4
−180	1. 0.180	+	+		+			1
	4. *0. 45*	+	+		+			e^2
	7. 3. 27	+	−		−			e
	7.–*3. 27*	+	−		−			e^3
	5. 0. 26	−	+		+			c
	9. *0. 20*	−	+		+			ce^2
	8.–2. 23	−	−		−			ce
	8. 2. 23	−	−		−			ce^3
−181	1. 0.181	+			+			1
	5. 2. 37	+			+			k^2
	13. 1. 14	+			+			k^4
	13.–1. 14	+			+			k^6
	5.–*2. 37*	+			+			k^8
	7. 1. 26	−			−			k
	10. 3. 19	−			−			k^3
	2. 1. 91	−			−			k^5
	10.–3. 19	−			−			k^7
	7.–1. 26	−			−			k^9
−182	1. 0.182	+	+			+		1
	9. 4. 22	+	+			+		g^2
	9.–*4. 22*	+	+			+		g^4
	3. 1. 61	−	+			−		g
	13. 0. 14	−	+			−		g^3
	3.–*1. 61*	−	+			−		g^5
	11.–4. 18	+	−			−		cg^2
	2. 0. 91	+	−			−		c
	11. *4. 18*	+	−			−		cg^4
	6. 2. 31	−	−			+		cg^5
	7. 0. 26	−	−			+		cg^3
	6.–2. 31	−	−			+		cg

D	Class	a	b	c	d	e	de	Cp
−183	1. 0.183	+	+					1
	13.–5. 16	+	+					i^2
	3. 0. 61	+	+					i^4
	13. *5. 16*	+	+					i^6
	8. 1. 23	−	−					i
	11.–2. 17	−	−					i^3
	11. 2. 17	−	−					i^5
	8.–1. 23	−	−					i^7
	2. 1. 92	+	+					σ
	4. 1. 46	−	−					σi
	8.–3. 24	+	+					σi^2
	12. 3. 16	−	−					σi^3
	6. 3. 32	+	+					σi^4
	12.–3. 16	−	−					σi^5
	8. 3. 24	+	+					σi^6
	4.–1. 46	−	−					σi^7
−184	1. 0.184	+			+	+		1
	8. *4. 25*	+			+	+		e^2
	5. 1. 37	−			+	−		e
	5.–*1. 37*	−			+	−		e^3
	11. 5. 19	−			−	−		ce
	11.–*5. 19*	−			−	−		ce^3
	4. 2. 47	+			−	+		c
	8. 0. 23	+			−	+		ce^2
−185	1. 0.185	+	+		+			1
	9.–2. 21	+	+		+			i^2
	10. 5. 21	+	+		+			i^4
	9. *2. 21*	+	+		+			i^6
	3. 1. 62	−	+		−			i
	7. 2. 27	−	+		−			i^3
	7.–2. 27	−	+		−			i^5
	3.–*1. 62*	−	+		−			i^7
	2. 1. 93	−	−		+			c
	13. 6. 17	−	−		+			ci^2
	5. 0. 37	−	−		+			ci^4
	13.–*6. 17*	−	−		+			ci^6
	6. 1. 31	+	−		−			ci
	14.–5. 15	+	−		−			ci^3
	14. 5. 15	+	−		−			ci^5
	6.–1. 31	+	−		−			ci^7
−186	1. 0.186	+	+				+	1
	10.–2. 19	+	+				+	g^2
	10. *2. 19*	+	+				+	g^4
	5. 2. 38	−	+				−	g
	2. 0. 93	−	+				−	g^3
	5.–*2. 38*	−	+				−	g^5
	11. 1. 17	−	−				+	cg^2
	3. 0. 62	−	−				+	c
	11.–*1. 17*	−	−				+	cg^4
	13.–3. 15	+	−				−	cg^5
	6. 0. 31	+	−				−	cg^3
	13. 3. 15	+	−				−	cg

D	Class	a	b	c	d	e	de	Cp
−187	1. 0.187	+	+					1
	4. 1. 47	+	+					g^2
	4.–*1. 47*	+	+					g^4
	7. 3. 28	−	−					g
	11. 0. 17	−	−					g^3
	7.–3. 28	−	−					g^5
	2. 1. 94	+	+					σ
	14. 3. 14	−	−					σg
−188	1. 0.188	+			+			1
	9. 1. 21	+			+			k^2
	12.–4. 17	+			+			k^4
	12. 4. 17	+			+			k^6
	9.–*1. 21*	+			+			k^8
	3. 1. 63	+			−			k
	7.–1. 27	+			−			k^3
	4. 0. 47	+			−			k^5
	7. 1. 27	+			−			k^7
	3.–1. 63	+			−			k^9
−189	1. 0.189	+	+		+			1
	9. 3. 22	+	+		+			g^2
	9.–*3. 22*	+	+		+			g^4
	5. 1. 38	−	−		+			g
	14. 7. 17	−	−		+			g^3
	5.–*1. 38*	−	−		+			g^5
	11.–3. 18	−	+		−			cg^2
	2. 1. 95	−	+		−			c
	11. *3. 18*	−	+		−			cg^4
	10. 1. 19	+	−		−			cg
	7. 0. 27	+	−		−			cg^3
	10.–1. 19	+	−		−			cg^5
−190	1. 0.190	+	+			+		1
	2. 0. 95	−	−			+		c_1
	5. 0. 38	−	+			−		c_2
	10. 0. 19	+	−			−		c_1c_2

D	Class	a	b	c	d	e	de	Cp
−191	1. 0.191	+						1
	3. 1. 64	+						n
	9. 4. 23	+						n^2
	8.−3. 25	+						n^3
	13.−2. 15	+						n^4
	5. 2. 39	+						n^5
	15. 7. 16	+						n^6
	15.−7. 16	+						n^7
	5.−2. 39	+						n^8
	13. 2. 15	+						n^9
	8. 3. 25	+						n^{10}
	9.−4. 23	+						n^{11}
	3.−1. 64	+						n^{12}
	2. 1. 96	+						σ
	6. 1. 32	+						σn
	12. 5. 18	+						σn^2
	4. 1. 48	+						σn^3
	12. 1. 16	+						σn^4
	10.−3. 10	+						σn^5
	8. 1. 24	+						σn^6
	8.−1. 24	+						σn^7
	10. 3. 10	+						σn^8
	12.−1. 16	+						σn^9
	4.−1. 48	+						σn^{10}
	12.−5. 18	+						σn^{11}
	6.−1. 32	+						σn^{12}
−192	1. 0.192	+			+	+		1
	4. *2. 49*	+			+	+		e^2
	7. 2. 28	+			−	+		e
	7.−*2. 28*	+			−	+		e^3
	13. 4. 16	+			+	−		ce
	13.−*4. 16*	+			+	−		ce^3
	3. 0. 64	+			−	−		c
	12. 6. 19	+			−	−		ce^2
−193	1. 0.193	+			+			1
	2. *1. 97*	+			+			e^2
	11. 4. 19	−			−			e
	11.−4. 19	−			−			e^3

D	Class	a	b	c	d	e	de	Cp
−194	1. 0.194	+					+	1
	11.−2. 18	+					+	y^2
	3. 1. 65	+					+	y^4
	6. 2. 33	+					+	y^6
	9.−2. 22	+					+	y^8
	2. 0. 97	+					+	y^{10}
	9. 2. 22	+					+	y^{12}
	6.−2. 33	+					+	y^{14}
	3.−1. 65	+					+	y^{16}
	11. *2. 18*	+					+	y^{18}
	5. 1. 39	−					−	y
	14.−4. 15	−					−	y^3
	13.−1. 15	−					−	y^5
	7.−3. 29	−					−	y^7
	10. 4. 21	−					−	y^9
	10.−4. 21	−					−	y^{11}
	7. 3. 29	−					−	y^{13}
	13. 1. 15	−					−	y^{15}
	14. 4. 15	−					−	y^{17}
	5.−1. 39	−					−	y^{19}
−195	1. 0.195	+	+	+				1
	4.−1. 49	+	+	+				g^2
	4. *1. 49*	+	+	+				g^4
	7. 1. 28	+	−	−				g
	13. 0. 15	+	−	−				g^3
	7.−*1. 28*	+	−	−				g^5
	12. 3. 17	−	−	+				cg^2
	3. 0. 65	−	−	+				c
	12.−*3. 17*	−	−	+				cg^4
	11.−5. 20	−	+	−				cg
	5. 0. 39	−	+	−				cg^3
	11. 5. 20	−	+	−				cg^5
	2. 1. 98	+	+	+				σ
	14. 1. 14	+	−	−				σg
	6. 3. 34	−	−	+				σc
	10. 5. 22	−	+	−				σcg
−196	1. 0.196	+			+			1
	8.−2. 25	+			+			i^2
	4. 0. 49	+			+			i^4
	8. *2. 25*	+			+			i^6
	5. 2. 40	−			+			i
	13. 5. 17	−			+			i^3
	13.−5. 17	−			+			i^5
	5.−2. 40	−			+			i^7

D	Class	a	b	c	d	e	de	Cp
−197	1. 0.197	+			+			1
	9. 1. 22	+			+			k^2
	6.−1. 33	+			+			k^4
	6. 1. 33	+			+			k^6
	9.−*1. 22*	+			+			k^8
	3. 1. 66	−			−			k
	11. 1. 18	−			−			k^3
	2. 1. 99	−			−			k^5
	11.−1. 18	−			−			k^7
	3.−1. 66	−			−			k^9
−198	1. 0.198	+	+			+		1
	9. *0. 22*	+	+			+		e^2
	9. 3. 23	−	+			+		e
	9.−*3. 23*	−	+			+		e^3
	2. 0. 99	−	−			−		c
	11. *0. 18*	−	−			−		ce^2
	13. 6. 18	+	−			−		ce
	13.−6. 18	+	−			−		ce^3
−199	1. 0.199	+						1
	5. 1. 40	+						j
	8.−1. 25	+						j^2
	7. 2. 29	+						j^3
	13.−3. 16	+						j^4
	13. 3. 16	+						j^5
	7.−2. 29	+						j^6
	8. 1. 25	+						j^7
	5.−1. 40	+						j^8
	2. 1.100	+						σ
	10. 1. 20	+						σj
	4.−1. 80	+						σj^2
	14.−5. 16	+						σj^3
	8. 3. 26	+						σj^4
	8.−3. 26	+						σj^5
	14. 5. 16	+						σj^6
	4. 1. 80	+						σj^7
	10.−1. 20	+						σj^8
−200	1. 0.200	+			+	+		1
	9. 4. 24	+			+	+		g^2
	9.−*4. 24*	+			+	+		g^4
	3. 1. 67	−			−	−		g
	8. 4. 27	−			−	−		g^3
	3.−*1. 67*	−			−	−		g^5
	11. 3. 19	+			−	−		cg^2
	4. 2. 51	+			−	−		c
	11.−*3. 19*	+			−	−		cg
	12.−2. 17	−			+	+		cg
	8. 0. 25	−			+	+		cg^3
	12. 2. 17	−			+	+		cg^5